DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS, AMHERST
ADVANCED EXAM — ALGEBRA. FALL 2017

Passing Standard: It is sufficient to do FIVE problems correctly, including at least ONE FROM EACH of the THREE parts.

Part I. Group Theory and Representation Theory

1. Let G be a non-Abelian group of order 21. Describe G in terms of generators and relations.

2. Let ρ be the permutation representation assoicated to the action of D_3 (dihedral of order 6) on itself by conjugation. Decompose the character of ρ into irreducible D_3-characters. Show your work.

3. Let G be a group acting *faithfully* on a set X of five elements, in other words if $gx = x$ for all $x \in X$ then $g = id$. There are two orbits of this G-action, one of size 2 and one of size 3. What are the possible groups? Justify your reasoning.
Hint: Map G to a product of symmetric groups.

Part II. Commutative Algebra

4. Let $R = \mathbf{Z}[\sqrt{-2}] = \{a + b\sqrt{-2} : a, b \in \mathbf{Z}\} \subset \mathbf{C}$. Show that R is a unique factorization domain.

5. (a) Show that $\mathbf{C} \otimes_{\mathbf{R}} \mathbf{C}$ and $\mathbf{C} \otimes_{\mathbf{C}} \mathbf{C}$ *are not* isomorphic as $\mathbf{R}$-modules.
(b) Show that $\mathbf{Q} \otimes_{\mathbf{Z}} \mathbf{Q}$ and $\mathbf{Q} \otimes_{\mathbf{Q}} \mathbf{Q}$ *are* isomorphic as $\mathbf{Q}$-modules.
Hint: What are $\mathbf{R}$-modules? What are $\mathbf{Q}$-modules?

6. Let R be an integral domain and M a finitely generated R-module. We say that M is *torsion-free* if for all $r \in R$ and $m \in M$, if $rm = 0$ then $r = 0$ or $m = 0$.
(a) Show that if M is a free R-module then M is torsion-free.
(b) Show that if R is a principal ideal domain and M is torsion-free, then M is a free R-module.
(c) Give an example of an integral domain R and a finitely generated R-module M such that M is torsion-free and M is *not* a free R-module. Justify your reasoning.

Part III. Field Theory and Galois Theory

7. (a) Prove that $f(x) = x^3 + 3x + 2$ and $g(x) = x^5 + 4x + 6$ are irreducible over $\mathbf{Q}$.
(b) Let $\alpha \in \mathbf{C}$ be a root of $f(x)$ and $\beta \in \mathbf{C}$ be a root of $g(x)$. Determine the degree of the field extension $\mathbf{Q}(\alpha, \beta)/\mathbf{Q}$. Justify your reasoning.

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8. Let K be a finite field of size 3^6.
(a) Show that there are exactly 696 elements $\alpha \in K$ such that $K = \mathbf{F}_3(\alpha)$.
(b) Determine the number of monic irreducible polynomials over $\mathbf{F}_3$ of degree 6.
Justify your reasoning!

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9. Let $\alpha \in \mathbf{C}$ be a root of $h(x) = x^6 + 3 = 0$. Show that $\mathbf{Q}(\alpha)/\mathbf{Q}$ is a Galois extension, and determine its Galois group. Show your work!
Hint: What are the roots of h?

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