COMPLEX ANALYSIS BASIC EXAM
UNIVERSITY OF MASSACHUSETTS, AMHERST
DEPARTMENT OF MATHEMATICS AND STATISTICS
AUGUST 2016

- Each problem is worth 10 points.
- Passing Standard: **Do 8 of the following 10 problems**, *and*
  - Master's level: 45 points with three questions essentially complete
  - Ph. D. level: 55 points with four questions essentially complete
- Justify your reasoning!

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1. Show that the function $\exp(-z^2)$ has a primitive on $\mathbf{C}$. You do **not** have to find the actual primitive.

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2. Show that

$$\int_{-\infty}^{\infty} \left(\frac{\sin x}{x}\right)^3 dx = \frac{3\pi}{4}.$$

Show the contour and prove all estimates you use. *Hint:* Consider the function $\dfrac{-2+3e^{iz}-e^{3iz}}{z^3}$.

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3. Let f be an analytic function on the open unit disk $D := \{z : |z| < 1\}$, such that $|f(z)| < 1$ for all $z \in D$ and that $f(0) = 0$. Show that for any fixed $0 < r < 1$, the series

$$\sum_{n=0}^{\infty} f(z^n)$$

converges uniformly for $|z| \le r$.

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4. (a) [5 points] Determine the type of every isolated singularity (including possibly at infinity) of the function

$$\frac{e^z}{1+z^2}.$$

Show your work.

(b) [5 points] Find the Laurent series of

$$\frac{1}{(1+z^2)(2+z^2)}$$

for $1 < |z| < \sqrt{2}$. Show your work.

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5. (a) [5 points] Determine all conformal maps ϕ from the open unit disk to itself, such that $\phi(0)$ lies on the real axis and that the arc on the unit circle with angle $0 \le \theta \le \pi/2$ is taken to the arc with angle $\pi/2 \le \theta \le 7\pi/6$. Show your work.

(b) [5 points] Determine the fractional linear transformation φ that takes the points $-1, 0, 1$ respectively to the points $1, i, -1$, and determine the image under φ of the upper half plane. Show your work.

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6. (a) [5 points] Construct a holomorphic function on $\mathbf{C}$ with real part $e^{-x}(x \sin y - y \cos y)$. Show your work.

(b) [5 points] Let $u(x, y)$ be a real-valued harmonic function on $\mathbf{R}^2$ such that $u(x, y)^2$ is also harmonic. Show that u is constant.

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7. Let U be an open subset of the complex plane. Suppose that $\{f_n\}$ is a sequence of analytic functions defined on U and converges uniformly on compact subsets of U to an analytic function f. Show that $\{f_n'\}$ converges uniformly on compact subsets of U to f'.

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8. Let $P_n(z) = \sum_{k=0}^{n} z^k/k!$. Given R, prove that P_n has no zeros in the disk of radius R for all n sufficiently large.

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9. Let C be the circle $\{z : |z| = 2\}$ traversed counter-clockwise. Compute $\displaystyle\int_C \frac{z^{2n}\cos(1/z)}{1 - z^n} dz$ for **all integers** $n \geq 2$. Show your work.

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10. Let $f(z)$ be an entire function and suppose that $|f(z)| \leq |\sin^3(z)|$ for all $z \in \mathbf{C}$. Prove that $f(z) = \lambda \sin^3(z)$ for some $\lambda \in \mathbb{C}$.
