DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS AMHERST

ADVANCED CALCULUS/LINEAR ALGEBRA EXAM

SEPTEMBER 2016

Do all 7 problems. **Show your work.**

Passing Standard:

- M.S. level: 60% with three questions essentially complete (including at least one from each part);
- Ph.D. level: 75% with two questions from each part essentially complete.

1. Linear Algebra

1. Let A be an $n \times n$ complex matrix such that $A^2 = A$.

(a) Show that A is similar to a diagonal matrix.

(b) Show that the trace of A is a non-negative integer.

2. Let $T : \mathbf{R}^n \to \mathbf{R}^n$ be a linear transformation. Prove that there exists an m such that the kernel of T^m intersects the image of T^m only at the origin $\mathbf{0}$.

3. Let A be a square matrix.

(a) Prove that if every row adds up to 1, then $\det(A - I) = 0$.

(b) If $\det(A - I) = 0$, does $\det A = 1$? Prove or disprove.

2. Advanced Calculus

4. Let $f(x, y) = xy + \int_0^y \sin(t^2)dt$.

(a) Compute $\nabla f(a, b)$.

(b) Show that $(0, 0)$ is a saddle point of $f(x, y)$.

5. Let $f, g : [0, 1] \to \mathbf{R}$ be continuous. Assume that $f(x) < g(x)$ for all $x \in [0, 1]$. Prove that

$$\int_0^1 f dx < \int_0^1 g dx.$$

(Note that the inequality is strict.)

6. Define a recursive sequence $\{a_n\}$ by:

$$a_1 = 5; \qquad a_{n+1} = \sqrt{3 + a_n}.$$

Give a careful proof that the sequence converges and determine its limit.

7. Consider the vector field $\mathbf{F}(x, y, z) = \langle y^2 z, 2y - e^z, \sin x \rangle$. Evaluate the flux integral

$$\oiint_S \mathbf{F} \cdot \mathbf{n} dS$$

where S is the boundary of the region bounded by the cylinder $x^2 + y^2 = 4$ and the planes $z = 1$ and $z = 8 - y$, with outward pointing normal vector.