DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS AMHERST
MASTER'S OPTION EXAM — APPLIED MATH
September 2015

Do 5 of the following questions. Each question carries the same weight. Passing level is 60% and at least two questions substantially correct.

1. [20 points] A simplistic model of a fishery reads

$$\dot{N} = rN(1 - \frac{N}{K}) - H$$

where H represents the effects of fishing.
(a) Show that the model can be written in dimensionless form as:

$$\frac{dx}{d\tau} = x(1 - x) - h$$

for suitably defined dimensionless x, τ and h.
(b) Show that a bifurcation occurs at a certain value h_c and classify this bifurcation.
(c) Plot the vector field for different values of h.
(d) Discuss the long term behavior of the fish population for $h < h_c$ and for $h > h_c$, giving some relevant biological interpretation.

2. [20 points] Consider the rabbit-sheep problem for $x > 0$ and $y > 0$:

$$\dot{x} = x(5 - x - 2y)$$

$$\dot{y} = y(4 - x - y)$$

(a) Find the fixed points.
(b) Classify their stability and sketch the phase plane.

(c) Explain why there can not be any limit cycles in this system.

3. [20 points] Consider the problem $u_t = u_{xx}$ with homogeneous Dirichlet boundary conditions in $(0, 1)$ and $u(x, 0) = x$. Solve the PDE by separating the variables, applying the boundary conditions and then the initial condition.

4. [20 points] Consider the wave equation $u_{tt} = c^2 u_{xx}$ and the diffusion equation $u_t = k u_{xx}$.
(a) Prove the uniqueness of the solution for the wave equation in $(0, l)$ with initial conditions $u(x, 0) = \phi(x)$, $u_t(x, 0) = \psi(x)$, and boundary conditions $u_x(0, t) = 0$, $u_x(l, t) = 0$, by means of the energy method.
(b) For the diffusion equation, prove the uniqueness of its solution with initial condition $u(x, 0) = \phi(x)$ and with homogeneous Dirichlet boundary conditions using the maximum principle.
(c) Prove the same thing as in (b), but now using the energy method.

5. [20 points] Solve the PDE,

$$u_t + [(1-u)u]_x = 0, \qquad (x \in R, t > 0).$$

for the two initial conditions given below. For each case, does the solution exist globally? Explain your answer, and plot the solutions for typical times.

a) $u_1(x,0) = \begin{cases} 0 & \text{if} \quad x \le 0 \\ x & \text{if} \quad 0 < x \le 1 \\ 1 & \text{if} \quad x > 1 \end{cases}$

b) $u_2(x,0) = \begin{cases} 1 & \text{if} \quad x \le 0 \\ 1-x & \text{if} \quad 0 < x \le 1 \\ 0 & \text{if} \quad x > 1 \end{cases}$

6. [20 points]

a) Solve the 2D Laplace's equation $\Delta u = 0$, in the exterior of a disk $(r > 1)$ with boundary condition $u(1, \theta) = 3 + 2\cos(4\theta) - \sin(2\theta)$, and the condition that u must be bounded as $r \to \infty$.

b) Solve the radially symmetric equation $\Delta u = 12$ on the domain $a < r < b$ in R^3, with vanishing boundary conditions.

7. [20 points] Consider a system of $x(t) \in R$ governed by

$$\frac{d^2x}{dt^2} + 2\alpha\frac{dx}{dt} + x - x^2 = 0, \qquad \text{constant } 0 < \alpha < 1.$$

a) Find the equilibrium points, and classify them by type and stability.

b) Draw the phase portrait in the $(x, \frac{dx}{dt})$ plane, and describe the qualitative behavior of the system.