ADVANCED CALCULUS/LINEAR ALGEBRA BASIC EXAM

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SEPTEMBER 2014

Complete 7 of the following 9 problems. Please show your work. The passing standards are:

- Master's level: 60% with three questions essentially complete (including one from each part);
- Ph.D. level: 75% with two questions from each part essentially correct.

Linear Algebra

(1) Let V_1, V_2 be vector subspaces of $\mathbb{R}^n$. Prove that

$$\dim(V_1 \cap V_2) \geq \dim V_1 + \dim V_2 - n.$$

(2) Let V be the vector space of all 3×3 matrices with real entries, and consider the linear transformation $T\colon V \to V$ given by $T(X) = AX + XA$, where

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Compute the determinant $\det T$.

(3) Let A be an $n \times n$ complex matrix. Prove that $A^n = 0$ if and only if $I_n - tA$ is invertible for all nonzero $t \in \mathbb{C}$.

(4) Let V be the subspace of $\mathbb{R}^4$ generated by the vectors

$$v_1 = (1, 2, 0, 1), \quad v_2 = (0, 1, 2, -1), \quad v_3 = (2, 0, 1, -1).$$

Let W be the subspace generated by

$$w_1 = (3, 1, 1, -1), \quad w_2 = (0, 2, 2, 1).$$

Find the dimension and a basis of $V \cap W$ and $V + W$.

Advanced Calculus

(5) Let S be the surface

$$z = x^2 + y^2, \ z \leq 1,$$

oriented so that the normal vector has positive z-coordinate, and let $\mathbf{F}$ be the vector field $(yz, -xz + \sin(z), e^{x^2+y^2})$. Compute the surface integral

$$\int_S \mathbf{F} \cdot d\mathbf{A}.$$

(6) Find, with proof, a real number C so that

$$\left| C - \int_0^1 \frac{\sin x}{x} dx \right| < .01.$$

(7) Consider a vector field $\mathbb{F}$ on $\mathbb{R}^3 \setminus \{0\}$ of the form

$$\mathbf{F} = g(\|\mathbf{x}\|)\mathbf{x},$$

where $g \colon (0, \infty) \to \mathbb{R}$ is a C^1 function. Show that for any closed curve C in $\mathbb{R}^3 \setminus \{0\}$, the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{s}$$

vanishes.

(8) Let $\{f_n\}$ be a sequence of continuously differentiable functions on $[a, b]$, and suppose that $f_n \to f$ pointwise, and $f_n' \to g$ uniformly on $[a, b]$. Show that
 (a) $f_n \to f$ uniformly, and
 (b) f is differentiable, and $f' = g$.

(9) (a) Show that, if $\{a_n\}$ is a nonnegative decreasing sequence, the series $\sum_{n=1}^{\infty} a_n$ converges if and only if

$$\sum_{n=1}^{\infty} 2^n a_{2^n} = a_1 + 2a_2 + 4a_4 + 8a_8 + \dots$$

converges.

 (b) Use part (a) to show that

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \qquad \text{and} \qquad \sum_{n=1}^{\infty} \frac{1}{n(\log n)^p}$$

converge if and only if $p > 1$.