Complex analysis qualifying exam

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Do 8 out of the following 10 questions.

Each question is worth 10 points. To pass at the Master's level it is sufficient to have 45 points with 3 questions essentially correct. To pass at the PhD level it is sufficient to have 55 points with 4 questions essentially correct.

Note: All answers should be justified carefully.

(1) (a) (2 points) Let Ω be an open subset of $\mathbb{C}$ and a an element of Ω. Let

$$f\colon \Omega \setminus \{a\} \to \mathbb{C}$$

be a holomorphic function. Define the residue of f at a.

(b) Let γ denote the circle with center the origin and radius 2, with the counterclockwise orientation. Compute the following contour integrals

i. (4 points)

$$\int_\gamma \frac{e^z}{(z^2+5z+4)^2} dz.$$

ii. (4 points)

$$\int_\gamma \frac{1}{z\cos(1/z)} dz.$$

(2) Compute the improper integral

$$\int_0^\infty \frac{x^2}{x^4+1} dx.$$

(3) Let $f\colon \mathbb{C} \to \mathbb{C}$ be a holomorphic bijection. Show that $f(z) = az + b$ for some $a, b \in \mathbb{C}$, $a \neq 0$.

(4) (a) (2 points) State Rouché's theorem.

(b) (4 points) Prove the following more precise version of the fundamental theorem of algebra. Let

$$f(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$$

be a monic polynomial of degree n with complex coefficients. Let A be the maximum of $|a_0|, |a_1|, \ldots, |a_{n-1}|$. Then f has n roots (counting multiplicities) in the open disk with center 0 and radius $R = A + 1$.

(c) (4 points) Let $f(z) = e^z + 3z^n$ where n is a positive integer. Show that $f(z)$ has n distinct roots in the open disk with center 0 and radius 1.

(5) Let ω_1, ω_2 be nonzero complex numbers such that $\omega_1/\omega_2 \notin \mathbb{R}$. Let f be a meromorphic function on $\mathbb{C}$ such that $f(z + \omega_1) = f(z)$ and $f(z + \omega_2) = f(z)$.

(a) (4 points) Show that if f is holomorphic then f is constant.

(b) (6 points) Suppose f is not constant and there exists $a \in \mathbb{C}$ such that f is holomorphic on the open set

$$\mathbb{C} \setminus \{a + n_1\omega_1 + n_2\omega_2 \mid n_1, n_2 \in \mathbb{Z}\}.$$

Show that f has a pole of order ≥ 2 at a. [Hint: Show that the residue of f at a equals zero.]

(6) Consider the Laurent series

$$\tan(z) = \sum_{n=-\infty}^{\infty} a_n z^n$$

that is valid in the annulus $\{z \in \mathbb{C} \mid \frac{\pi}{2} < |z| < \frac{3\pi}{2}\}$. Use contour integrals to determine the coefficients a_n with indices in the range $-\infty < n \leq -1$.

(7) Let $D = \{z \in \mathbb{C} \mid |z| < 1\}$ be the open unit disk and $f\colon D \to D$ a holomorphic map.

(a) Prove the inequality $|f'(z)| \leq \dfrac{1 - |f(z)|^2}{1 - |z|^2}$ for all z in D.

(b) Let $\gamma : [0, 1] \to D$ be a smooth path in D. The integral

$$\int_\gamma \frac{|dz|}{1 - |z|^2}$$

is called the hyperbolic length of γ. Show that the hyperbolic length of $f \circ \gamma$ is less than or equal to that of γ. Deduce that the hyperbolic lengths are equal if f is a holomorphic automorphism of D.

(8) State and prove the Casorati–Weierstrass Theorem concerning the properties of a function in a neighborhood of an isolated singularity.

(9) Show that a single valued holomorphic branch of $\sqrt{1 - z^2}$ can be defined in any connected open subset U of the complex plane such that the points 1 and -1 are in the same connected component of the complement of U.

(10) Set

$$U := \{z \in \mathbb{C} \mid 0 < \mathrm{Im}(z) < \pi\}$$

(an infinite strip of height π), and

$$V := \{z \in \mathbb{C} \mid |z| < 1 \text{ and } \mathrm{Im}(z) > 0\}$$

(the upper half of the open unit disk). Find a bijective conformal mapping from U to V.