Department of Mathematics and Statistics
University of Massachusetts

Basic Exam: Topology

August 29, 2011

Answer five of the seven questions. Indicate clearly which five questions you want graded. Justify your answers.
Passing standard: For Master's level, 60% with two questions essentially complete. For Ph.D. level, 75% with three questions essentially complete. **A problem appears on the back of this page.**

Throughout this exam, $\mathbb{R}$ denotes the real line with the standard topology.

(1) Let X, Y be topological spaces with Y compact. Let $p\colon X \times Y \to X$ be projection onto the first factor. Show that p maps each closed set in $X \times Y$ to a closed set in X.

(2) Give an example of a locally connected space X and a continuous surjective map $f : X \to Y$ such that Y is not locally connected.

(3) Let $X = [0, 3]/(1, 2)$ be the quotient space of the closed interval $[0, 3]$ with the open interval $(1, 2)$ identified to a point. Let $f : X \to \mathbb{R}$ be a continuous map.
  (a) Prove that f achieves a global minimum.
  (b) Prove that X is not Hausdorff.

(4) Let $X \subset \mathbb{R}^n$ be a subspace. Let $F : \mathbb{R}^n \to Y$ be a continuous map and $f = F|_X$ be the restriction. Show that the induced homomorphism

$$f_* : \pi_1(X, x) \to \pi_1(Y, f(x))$$

is trivial (sends everything to the identity element).

(5) Let $\{A_\alpha\}$ be a collection of subsets of X such that $X = \bigcup A_\alpha$. Let $f\colon X \to Y$ and suppose that the restrictions $f|_{A_\alpha}$ are all continuous.
  (a) Show that if the collection $\{A_\alpha\}$ is finite and each A_α is closed, then f is continuous.
  (b) Find an example where the collection is countably infinite and each A_α is closed, but f is not continuous.

(6) Recall that $g\colon X \to Y$ is called *proper* if $g^{-1}(C)$ is compact whenever $C \subset Y$ is compact. Show that if a (not necessarily continuous) map $f\colon X \to Y$ is closed and $f^{-1}(y)$ is compact for all $y \in Y$, then f is proper.

(7) Let X be the infinite 3-valent tree. Thus X is an infinite graph containing no cycle, and such that every vertex is incident to three edges. Let x_0 be a fixed vertex. Give X a metric d by identifying each edge and its two vertices with the closed interval $[0, 1] \subset \mathbb{R}$ endowed with the usual Euclidean metric, and give X the metric topology. For each nonnegative integer n let X_n be the closed subset $\{x \in X \mid d(x, x_0) \leq n\}$. (See the figure for a picture of X_3.)

(a) Prove that each X_n is compact.
(b) Prove that each X_n is contractible.
(c) Prove that X is simply-connected.

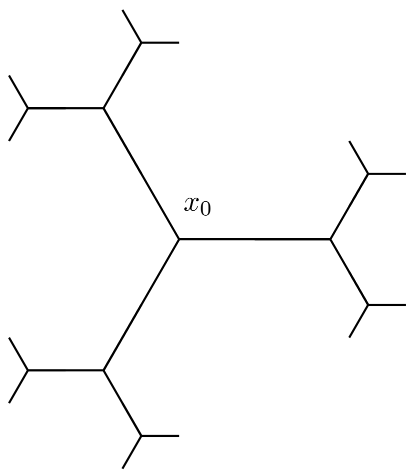

FIGURE 1. The subset X_3 of the infinite 3-valent tree.