Department of Mathematics and Statistics
University of Massachusetts

Basic Exam - Complex Analysis

August 2011

Do eight out of the following 10 questions. Each question is worth 10 points. To pass at the Master's level it is sufficient to have 45 points, with 3 questions essentially correct; 55 points with 4 questions essentially correct are sufficient for passing at the Ph.D. level.

Note: All answers should be justified.

1. Compute the integral
$$\int_0^\infty \frac{x \sin 3x}{(x^2+1)(x^2+4)} dx.$$
Justify your answer carefully.

2. Consider the meromorphic function
$$f(z) = \frac{1}{z^2 - (2+5i)z + 10i}.$$
In each of the following cases, compute the Laurent series
$$f(z) = \sum_{n \in \mathbb{Z}} c_n (z-a)^n$$
of f centered at a which is valid in a neighbourhood of b, and determine its domain of convergence
   (i) $a = b = 2$.
   (ii) $a = 0$, $b = 3$.

3. For each $n \geq 3$, find the number of zeros (counting multiplicities) of $z^n + 3z + 1$ in the annulus $A = \{1 < |z| < 2\}$. Determine whether these zeros are all simple.

4. Let U be the portion of the open unit disk given in polar coordinates by
$$U \ := \ \{re^{i\theta} \ : \ 0 < r < 1, \text{ and } 0 < \theta < \pi/3\}.$$
The the boundary of U consists of the line segment L_0 from 0 to 1, the line segment $L_{\pi/3}$ from 0 to $e^{\pi i/3}$, and a curve Γ on the unit circle. Prove that there exists a unique fractional linear transformation f satisfying $f(1) = i$, $f(e^{\pi i/3}) = 0$, f maps Γ into the imaginary line $\mathbb{R}i$, and f maps $L_{\pi/3}$ into the real axis. Give an explicit, simple formula for $f(z)$. Justify your answer. Hint: Find $f^{-1}(\infty)$ first.

5. Show that if $f(z)$ is meromorphic in the extended complex plane $\mathbb{C} \cup \{\infty\}$ then f is a rational function.

6. State and prove Liouville's Theorem for entire functions.

7. Evaluate the following integrals, where $C = \{z \ : \ |z| = 4\}$ traversed once in the counterclockwise direction.

   (a) $\displaystyle\int_C \frac{z^4}{e^z + 1} dz.$

   (b) $\displaystyle\int_C \frac{z^3 \cos(1/z)}{z^4 + 1} dz.$

8. Prove that the series

$$\sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}$$

defines an analytic function $f(z)$ in the open set $= \mathbb{C} \setminus \mathbb{Z}$. Prove also that f has an antiderivative on U.

9. Calculate

$$\int_0^{\pi} \frac{dx}{2 + \cos^2(x)}.$$

10. Let f be a one-to-one holomorphic map from a region Ω_1 onto a region Ω_2. Assume that the closure of the unit disc $D := \{z \ : \ |z| < 1\}$ is contained in Ω_1. Prove that the inverse function $f^{-1} : f(D) \to D$ is given by the integral formula

$$f^{-1}(\omega) = \frac{1}{2\pi i} \int_{\{|z|=1\}} \frac{f'(z)}{f(z) - \omega} \cdot z \, dz$$