UNIVERSITY OF MASSACHUSETTS
Department of Mathematics and Statistics
ADVANCED EXAM - Mathematical Statistics and Probability
Thursday, September 2, 2010

Work all problems. 70 points are required to pass with at least 25 from each part (the Mathematical Statistics part consists of problems 1 - 3 and the Probability part consists of problems 4-6).

Part I: Mathematical Statistics

1. (15 PTS) Suppose $\mathbf{X}' = [X_1, X_2, X_3]$ has $\mathbf{N_3}(\mu, \mathbf{\Sigma})$ distribution, where

$$\mu = \begin{pmatrix} -2 \\ 3 \\ 5 \end{pmatrix}, \quad \mathbf{\Sigma} = \begin{bmatrix} 4 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix}$$

   (a) Find the joint distribution of X_2 and X_3.

   (b) Find the correlation coefficient ρ_{23} of X_2 and X_3.

   (c) Find the conditional distribution of X_1 given $X_2 = x_2$ and $X_3 = x_3$.

   (d) Find the distribution of Y, where $Y = 4X_1 - 6X_2 + X_3 - 18$.

   (e) Referring to the previous problem, find a constant c such that $\Pr[Y > c] = 0.95$.

   (f) Find the joint distribution of Z_1, Z_2, where

$$\begin{aligned} Z_1 &= X_1 - X_2 + 2X_3 - 6 \\ Z_2 &= 2X_2 + 4 \end{aligned}$$

2. (20 PTS)

   (a) Chris is working with two random variables (X,Y) and believes $X \mid Y = y \sim N(ry, 1 - r^2)$ and $Y \mid X = x \sim N(rx, 1 - r^2)$. Chris knows nothing else about them, including the value of r. Chris tries to determine the joint distribution by writing the following code in R:

```
n.tries <- 100000
sim.r <- function(r){
  x <- rep ( NA, n.tries )
  y <- rep ( NA, n.tries )
  x[1] <- 0
```

```
    y[1] <- 0
    for ( i in 2:n.tries ) {
      x[i] <- rnorm ( 1, r*y[i-1], sqrt(1-r^2) )
      y[i] <- rnorm ( 1, r*x[i], sqrt(1-r^2) )
    }
    return ( data.frame(x=x,y=y) )
  }
```

i. Are there two random variables having those conditional distributions?

ii. Is the joint distribution of (X,Y) uniquely determined by those conditional distributions?

iii. Chris runs the code for many values of r in the open interval (0,1). How can Chris use the results to learn the joint distribution of (X,Y)?

iv. Chris runs the code for r=1. What happens?

(b) Chris runs the code for $X \mid Y = y \sim N(y, 1)$ and $Y \mid X = x \sim N(x, 1)$. What happens?

3. (15 PTS) Let $(X_1, Y_1), \ldots, (X_n, Y_n)$ be independent copies of (X, Y), whose joint distribution is specified as follows: the marginal distribution of X is Poisson(λ), and conditioning on $X = x$, Y is distributed as binomial($x + 1$, p).

   (a) Show that the covariance between X and Y is $\alpha = p\lambda$.

   (b) Find the maximum likelihood estimate of α, say $\hat{\alpha}$.

   (c) Derive the asymptotic distribution of $\sqrt{n}(\hat{\alpha} - \alpha)$.

Part II: Probability

1. (10 PTS)
   (a) Give a precise definition for convergence in distribution
   (b) Give a precise definition for convergence in probability
   (c) Give a precise definition for convergence almost surely
   (d) Give examples of sequences of random variables that converge in distribution but not in probability
   (e) Give examples of sequences of random variables that converge in probability but not almost surely

2. (20 PTS) Let $\{\xi_n\}$ be a sequence of random variables on the probability space $(\Omega, \mathcal{F}, \mathcal{P})$ and $p \in (0, \infty)$.
   (a) If $\xi_n \to 0$ in $L^p(\Omega, P)$ show that $\xi_n \to 0$ in probability.
   (b) If $\xi_n \to 0$ in probability and there exists $\eta \in L^p(\Omega, P)$ such that $|\xi_n| < \eta$, show that $\xi_n \to 0$ in $L^p(\Omega, P)$.

3. (20 PTS)
   (a) Assume $\xi_n, n = 1, 2, \cdots$ are independent random variables on the probability space $(\Omega, \mathcal{F}, \mathcal{P})$ such that $Var(\xi_n) \leq C < \infty$. Formulate and prove a version of the weak law of large numbers for this sequence of random variables.
   (b) Let $f(x)$ be a continuous function on [a,b]. Construct an algorithm to estimate $\int_a^b f(x)dx$ by counting random dots and use the Strong Law of Large Numbers to prove that it converges.