DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS AMHERST
MASTER'S OPTION EXAM — APPLIED MATH
August 2009

Do 5 of the following questions. Each question carries the same weight. Passing level is 60% and at least two questions substantially correct.

1. Consider the linear ODE

$$2x\frac{dy}{dx} + y \; = \; f(x)\,, \qquad \text{for } x \geq 0\,,$$

in which f is a smooth and strictly negative function. Show that it is not possible for any solution $y(x)$ to have a *finite and positive* initial value $y(0)$. How do the positive solutions behave as x approaches 0?

2. A nonlinear oscillator with displacement $x \in R^1$ is governed by the DE:

$$\frac{d^2x}{dt^2} + \frac{dV}{dx} = 0\,, \qquad \text{with potential} \quad V = \frac{1}{2}x^2 - \frac{1}{3}x^3\,.$$

(a) Reformulate this dynamical equation as a two-dimensional system of first-order equations. Determine the equilibrium points of the system.

(b) Analyze the stability of each equilibrium point and sketch the entire phase portrait.

(c) Find a function $H = H(x, \dot{x})$ on the phase plane that is constant on each solution trajectory.

3. Consider the competing species model governing the evolution of two ecological species quantified by x_1 and x_2 (with $x_1, x_2 \geq 0$):

$$\begin{aligned}\frac{dx_1}{dt} &= r_1\left(1 - \frac{x_1}{k_1}\right)x_1 - c_1 x_1 x_2\\ \frac{dx_2}{dt} &= r_2\left(1 - \frac{x_2}{k_2}\right)x_2 - c_2 x_1 x_2\,.\end{aligned}$$

(a) Interpret the positive constants r_i, k_i and c_i ($i = 1, 2$), and describe the meaning of the model's terms that are scaled by these constants.

(b) In the case of "weak competition" when $c_1 < r_1/k_2$ and $c_2 < r_2/k_1$, determine the qualitative behavior of solutions as $t \to +\infty$? What does this behavior mean in ecological terms? Do find the nullclines and equilibrium point(s) and sketch them, but do not carry out a full stability analysis of the equilibrium point(s) [because the algebra is too messy.]

4. (a) Provide a complete derivation of the Laplace operator in polar coordinates in R^2. That is, show that the operator $u \mapsto \triangle u = \partial^2 u/\partial x^2 + \partial^2 u/\partial y^2$ converts to

$$u \mapsto \triangle u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\,.$$

(b) Solve the following boundary value problem:

$$\begin{aligned}\triangle u &= 0 \qquad \text{in} \quad 0 \leq r < a,\ \ 0 \leq \theta < 2\pi\,,\\ u &= \sin^2\theta \qquad \text{on} \quad r = a\,.\end{aligned}$$

5. The probability density function (PDF) $u(x,t)$ for an elastically bound particle evolves according to the equation

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + \gamma\frac{\partial(xu)}{\partial x},$$

for $-\infty < x < \infty$ and $t > 0$, where D and γ are positive constants. Verify that for all $t > 0$, the solution $u(x,t)$ is a PDF provided the data $u(x,0)$ is. A function $v(x)$ is a PDF if and only if it satisfies *both* conditions

$$v(x) \geq 0 \quad \text{and} \quad \int_{-\infty}^{+\infty} v(x)\,dx = 1.$$

6. The "shallow water equations" approximate the motion of a thin layer of incompressible and inviscid fluid in the presence of gravity. In one space dimension they are the following pair of nonlinear PDEs:

$$\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} = -h\frac{\partial u}{\partial x}$$
$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -g\frac{\partial h}{\partial x}\,.$$

for a pair of unknowns, $h = h(x,t)$, $u = u(x,t)$ that represent the water surface height and the fluid velocity (in the x direction). The constant g is the gravitational acceleration.

(a) Consider motions that are small perturbations around the uniform, undisturbed state $h = H, u = 0$, where H is a constant water height. In terms of the perturbation variables $\eta \doteq h - H$ and u, derive the linearized equations of motion (in which terms of higher order than the first in the perturbations are neglected).

(b) Show that this pair of linear first-order PDEs in the variables $\eta \doteq h - H$ and u are equivalent to a single, second-order PDE in η; namely, the wave equation

$$\frac{\partial^2\eta}{\partial t^2} - c^2\frac{\partial^2\eta}{\partial x^2} = 0\,.$$

Give a formula for the wave speed c in terms of g and H.

7. The viscous Burgers' equation for $u(x,t)$,

$$u_t + \left(\frac{1}{2}\,u^2\right)_x = \epsilon\, u_{xx}, \qquad (\, x \in R^1,\ t > 0\,)$$

is a fundamental equation for nonlinear viscous flows.

(a) Make the substitution

$$w(x,t) = \int_{-\infty}^{x} u(\xi,t)\,d\xi,$$

and derive the PDE satisfied by $w(x,t)$.

(b) Now make the substitution

$$w(x,t) = \alpha\,\log\phi(x,t)\,, \qquad \text{where } \alpha \text{ is a positive constant,}$$

and derive an equivalent PDE for ϕ.

(c) Conclude that for an appropriate choice of constant α, solutions u of Burgers' equation are in 1-1 correspondence with solutions ϕ of the heat equation. This is known as the "Cole-Hopf transformation."