Department of Mathematics and Statistics
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Basic Exam: Topology

August 27, 2008

Answer five of the seven questions. Indicate clearly which five questions you want graded. Justify your answers.

Passing standard: For Master's level, 60% with two questions essentially complete. For Ph.D. level, 75% with three questions essentially complete.

Let $\mathbb{R}$ denote the real line with the standard topology.

(1) Let X be a set endowed with two metrics d_1, d_2, and let $\mathcal{T}_1$, $\mathcal{T}_2$ be the metric topologies on X attached to these metrics. Suppose that these topologies have the following property: for any $p \in X$ and any open ball B_1 in $\mathcal{T}_1$ **with center** p, there is an open ball B_2 in $\mathcal{T}_2$ **with center** p such that $B_2 \subset B_1$. Show that $\mathcal{T}_1$ is coarser than $\mathcal{T}_2$.

(2) Let S^1 be the unit circle in $\mathbb{R}^2$. Equip $S^1 \times S^1$ with the product topology. Let $f : S^1 \times S^1 \to (0, 1)$ be continuous.

(a) Prove that f cannot be surjective.

(b) Prove that there exists an $x \times y \in S^1 \times S^1$ such that

$$f(x \times y) = f((-x) \times y).$$

(3) Let X be the union of the x- and y-axes in $\mathbb{R}^2$,

$$X = (\mathbb{R} \times \{0\}) \cup (\{0\} \times \mathbb{R}) \subset \mathbb{R} \times \mathbb{R}$$

Give $X \subset \mathbb{R}^2$ the subspace topology. Define the map $g : \mathbb{R}^2 \to X$ by the equations

$$\begin{aligned} g(x \times y) &= x \times 0, \quad \text{if } x \neq 0 \\ g(0 \times y) &= 0 \times y, \quad \text{otherwise} \end{aligned}$$

(a) Prove that g is not a quotient map.

(b) Show that in the quotient topology induced by g, the space X is not Hausdorff.

(4) Prove or disprove the following:

(a) If X and Y are path-connected, then $X \times Y$ is path-connected.

(b) If $A \subset X$ is path-connected, then its closure $\overline{A}$ is path-connected.

(c) If X is path-connected and $f : X \to Y$ is continuous, then $f(X)$ is path-connected.

(d) If $f\colon X \to Y$ is continuous and $f(X)$ is path-connected, then X is path-connected.

(5) Consider the sequence of continuous functions $f_n : \mathbb{R} \to \mathbb{R}$ given by $f_n(x) = e^{-nx^2}$.

(a) Prove that $\{f_n\}$ converges in the point-open topology (pointwise convergence) but not in the compact-open topology (compact convergence), nor in the uniform topology.

(b) Construct a continuous function $g : \mathbb{R} \to \mathbb{R}$ such that the sequence $g_n = f_n \circ g$ converges in the compact-open topology but not the uniform topology. Represent the function either algebraically, or with a clear picture. Justify your answer.

(6) (a) Let $f : X \to Y$ be a closed continuous surjection. Prove that if Y is compact and $f^{-1}(y)$ is compact for every $y \in Y$, then X is compact.

(b) Prove by example that the above statement is false if the surjective hypothesis is dropped.

(7) Prove that a compact connected Hausdorff space either has one point or cannot be countable.