DEPARTMENT OF MATHEMATICS AND STATISTICS
UMASS - AMHERST
BASIC EXAM - PROBABILITY
August 2008

Work all problems. Show all work. Explain your answers. State the theorems used whenever possible. 60 points are needed to pass at the Masters Level and 75 to pass at the Ph.D. level.

1. Let X have the double exponential distribution with density $f(x) = \frac{1}{2}\exp(-|x|)$ for $-\infty < x < \infty$.
   (a) (6pts) Find the moment generating function (MGF) of X.
   (b) (6pts) Compute the mean and variance of X using the MGF obtained in (a).
   (c) (6pts) Let $X_1, \cdots, X_{100}$ be independent random variables, all of which have the double exponential distribution. Let $\bar{X} = \frac{1}{100}\sum_{i=1}^{100} X_i$. Find δ such that $P(|\bar{X}| < \delta) \approx 0.95$.
   (d) (6pts) Find the distribution of $Y = |X|$.

2. Let X_1 and X_2 be independent, identically distributed random variables that have the common probability density function: $f(x) = \exp(-x), x > 0$. Let $Y_1 = X_1 + X_2$ and $Y_2 = X_1 - X_2$.
   (a) (6pts) Find the joint pdf of Y_1 and Y_2.
   (b) (6pts) Find the marginal pdf of Y_1.
   (c) (6pts) Find the conditional pdf of Y_2, given that $Y_1 = y_1$.

3. The hourly number of phone calls, X, received by a switchboard at a specific company is Poisson with parameter λ. λ varies independently from hour to hour according to the following exponential distribution: $f(\lambda) = \theta\exp(-\theta\lambda), \lambda > 0$. Answers to the following questions may depend on θ.
   (a) (7pts) Find an integral expression for $P(X = 4|\lambda \leq 6)$. You do not need to evaluate the integral.
   (b) (7pts) Find $E(X|\lambda)$ and its distribution.
   (c) (7pts) Use the preceding part to calculate $E(X)$.
   (d) (7pts) Assuming that the number of phone calls to the switchboard is independent from hour to hour, how many hours would be expected to have exactly 3 phone calls during a 24-hour period?

4. Let $X_1, \cdots, X_n$ be iid with density $f(x) = \beta(1-x)^{\beta-1}, 0 < x < 1$. Define $X_{(n)} = \max_{1\le i\le n} X_i$.

(a) (7pts) Find $P(n^{\nu}(1 - X_{(n)}) < x)$ for any fixed ν and $x \in (0,1)$.

(b) (9pts) State the definition of convergence in distribution and find a value of ν so that $n^{\nu}(1 - X_{(n)})$ converges in distribution.

(c) (7pts) Let $\bar{X}_n = n^{-1}\sum_{i=1}^{n} X_i$. Show that $\sqrt{n}(\bar{X}_n - 1/(1+\beta))$ converges in distribution and say what it converges to.

(d) (7pts) Let $T_n = \bar{X}_n^2$. Find an approximate distribution of T_n as n goes to infinity.