Department of Mathematics and Statistics
University of Massachusetts

Basic Exam - Complex Analysis

August 2008

Do eight out of the following 10 questions. Each question is worth 10 points. To pass at the Master's level it is sufficient to have 45 points, with 3 questions essentially correct; 55 points with 4 questions essentially correct are sufficient for passing at the Ph.D. level.

Note: All answers should be justified.

1. Use the Residue Theorem to prove the equality

$$I = \int_0^\pi \frac{\cos(4\theta)}{1+\cos^2(\theta)} d\theta \quad = \quad -12\pi + \pi \cdot 17/\sqrt{2}.$$

2. Suppose f is holomorphic in the punctured disc $0 < |z| < R$, with associated Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n.$$

Suppose there is a positive real number M such that

$$r^4 \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta < M, \qquad \text{for all } 0 < r < R.$$

Prove that $a_n = 0$ for $n < -2$.

3. Let f be a complex-valued function on the open unit disk and m, n two positive integers which have greater common divisor $\gcd(m, n) = 1$. Assume that $g = f^n$ and $h = f^m$ are both holomorphic on the open unit disk. Prove that f is holomorphic on the open unit disk.

4. Suppose f is entire and that $f(z)/z \to 0$ as $|z| \to \infty$. Prove that f is a constant function.

5. Fix a complex number τ with $Im(\tau) > 0$.

   (a) Show that the series

   $$f(z) \;=\; \sum_{n=-\infty}^{\infty} e^{\pi i[n^2\tau + 2nz]}$$

   defines a holomorphic function on the whole of $\mathbb{C}$.

   (b) Show that $f(z+\tau) = e^{-\pi i[\tau+2z]} f(z)$.

6. Let a, b be positive real numbers. Compute the real improper integral

$$\int_0^\infty \frac{\cos(ax)}{x^2+b^2} dx$$

Justify all your steps and prove all estimates.

7. Find a one-to-one conformal map f from the region

$$U \ := \ \mathbb{C} \setminus \mathbb{R}_{\leq 0} = \{z \ : Im(z) \neq 0 \text{ or } Re(z) > 0\}$$

onto the region

$$V \ := \ \{z \ : \ |z| < 1 \text{ and } |z - \frac{1}{2}| > \frac{1}{2}\}$$

inside the open unit disk and outside the closed disk of radius 1/2 centered at 1/2. Justify your answer.

8. Let $\mathbb{H} := \{z \ : \ Im(z) > 0\}$ be the upper-half-plane and $f : \mathbb{H} \to \mathbb{H}$ a one-to-one and onto holomorphic map. Use the Schwarz Lemma to prove that there exist real numbers a, b, c, d, with $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} > 0$, such that $f(z) = \frac{az+b}{cz+d}$, for all $z \in \mathbb{H}$. (State, but do not prove the Schwarz Lemma).

9. (a) Find the number of roots of the polynomial

$$P(z) = 2z^7 + 2z^6 - z^3 + z + 100$$

in the disk $\{z \ : \ |z| < 2\}$.

(b) Let C be the circle $|z| = 2$ oriented counterclockwise. Compute the integral $\int_C \frac{z^7 dz}{P(z)}$.

10. Prove or disprove the following statements. Please include the full and precise statement of any theorem you use.

(a) There exists a function f, holomorphic on the unit disk, and satisfying the inequalities

$$3 - 2x^2 - 2y^2 \ < \ |f(x+iy)|^2 \ < \ 3 - x^2 - y^2.$$

(b) Let U be a bounded open subset of the complex plane $\mathbb{C}$. Then the following are equivalent

i. For every holomorphic function f on U there exists a holomorphic function F on U, such that $F' = f$.

ii. There exists a one-to-one holomorphic map from U onto the unit disk.

(c) For every open subset U of the complex plane, and for every real valued function u, defined and harmonic on U, there exists a harmonic function v on U, such that $f(x+iy) = u(x,y) + iv(x,y)$ is a holomorphic function on U.