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Advanced Analysis Qualifying Examination
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Instructions

1. This exam consists of eight (8) problems all counted equally for a total of 100%.
2. You are encouraged to try to solve every problem; there is no penalty for incorrect answers.
3. In order to pass this exam, it is enough that you solve essentially correctly at least five (5) problems and that you have an overall score of at least 65%.
4. State explicitly all results that you use in your proofs and verify that these results apply.
5. Please write your work and answers clearly in the blank space under each question.

Conventions

1. For a set A, 1_A denotes the indicator function or characteristic function of A.
2. If a measure is not specified, use Lebesgue measure on $\mathbb{R}$. This measure is denoted by m.
3. If a σ-algebra on $\mathbb{R}$ is not specified, use the Borel σ-algebra $\mathcal{B}_{\mathbb{R}}$

1. Let $X = \mathbb{N} \cup \{0\}$, where $\mathbb{N}$ is the set of positive integers; let $\mathcal{M}$ be the class of all subsets of X; and for $E \in \mathcal{M}$ let $\mu(E)$ equal the number of points in E. The four parts of this problem refer to this measure space $(X, \mathcal{M}, \mu)$.

   (a) Let $\{f_n, n \in \mathbb{N}\}$ be a sequence of measurable functions mapping X into $\mathbb{R}$ and let f be a measurable function mapping X into $\mathbb{R}$. Prove that $\{f_n\}$ converges in measure to f if and only if $\{f_n\}$ converges to f uniformly.

   (b) Given $f \in L^1(\mu)$ and $\lambda > 0$, define $g(\lambda) = \sum_{j=0}^{\infty} \exp(-j\lambda) f(j)$. Evaluate $\lim_{\lambda \to \infty} g(\lambda)$ and justify your answer.

   (c) Define $h(j) = 2^{-j}$ for $j \in X$. Calculate $\int_X h d\mu$.

2. Let m denote Lebesgue measure on $[0, 1]$. This problem has two parts.

(a) Define A_4 to be the set of all $x \in [0, 1]$ having a decimal expansion $x = \sum_{j=1}^{\infty} a_j 10^{-j}$ such that $a_j \neq 4$ for all $j \in \mathbb{N}$. Compute $m(A_4)$.

(b) Define B to be the set of all $x \in [0, 1]$ having a decimal expansion $x = \sum_{j=1}^{\infty} b_j 10^{-j}$ such that for all $n \in \mathbb{N}$ satisfying $0 \leq n \leq 9$ there exists at least one $j \in \mathbb{N}$ for which $b_j = n$. Compute $m(B)$.

3. Let $(X, \mathcal{M}, \mu)$ be a measure space, $A \in \mathcal{M}$ satisfying $\mu(A) < \infty$, and f a nonnegative measurable function on A. Prove that the following three statements are equivalent.

   (a) f is integrable on A.

   (b) $\sum_{k=0}^{\infty} k\mu(A_k) < \infty$, where $A_k = \{x \in A : k \leq f(x) < k+1\}$.

   (c) $\sum_{k=0}^{\infty} \mu(B_k) < \infty$, where $B_k = \{x \in A : f(x) \geq k\}$.

4. Let m denote Lebesgue measure on $[0,1]$. Let $\{h_n, n \in \mathbb{N}\}$ be a sequence of measurable functions on $([0,1], \mathcal{B}_{[0,1]}, m)$ and h a measurable function on $([0,1], \mathcal{B}_{[0,1]}, m)$. We say that h_n converges to h in distribution, and write $h_n \xrightarrow{d} h$, if for every bounded continuous function φ

$$\int_{[0,1]} (\varphi \circ h_n) dm \to \int_{[0,1]} (\varphi \circ h) dm,$$

where $\circ$ denotes composition. This problem has two parts.

(a) Prove that if $h_n \to h$ a.e., then $h_n \xrightarrow{d} h$.

(b) Assume that there exists $K < \infty$ such that $|h_n| \leq K$ for all $n \in \mathbb{N}$ and $|h| \leq K$. Prove that if $h_n \to h$ in measure, then $h_n \xrightarrow{d} h$. (**Hint.** A bounded continuous function φ is uniformly continuous on $[-K, K]$).

5. Let $[a, b]$ be a closed bounded interval in $\mathbb{R}$. This problem has two parts.

(a) Let $f : [a, b] \to \mathbb{R}$ be a function of bounded variation, I an interval in $\mathbb{R}$ containing the range of f, and $g : I \to \mathbb{R}$ a Lipschitz-continuous function on I. Prove that the composition $g \circ f$ is of bounded variation. Recall that g is Lipschitz-continuous on an interval I if there exists $M < \infty$ such that for all x and y in I, $|g(x) - g(y)| \leq M|x - y|$.

(b) Let $f : [a, b] \to \mathbb{R}$ be a function of bounded variation for which there exists $\alpha > 0$ such that $f(x) \geq \alpha > 0$ for all $x \in [a, b]$. Prove that there exist two nondecreasing functions φ and ψ on $[a, b]$ such that

$$f(x) = \frac{\varphi(x)}{\psi(x)} \text{ for all } x \in [a, b].$$

(**Hint.** Apply part (a).)

6. For $x \in \mathbb{R}$ and $t > 0$ define

$$f(x,t) = \frac{1}{\sqrt{2\pi t}} \exp(-x^2/2t).$$

For all $t > 0$, $f(x,t)$ is normalized in the sense that $\int_{-\infty}^{\infty} f(x,t)dx = 1$. You do not need to prove this. For $x \in \mathbb{R}$ and $t > 0$ define

$$g(x,t) = \frac{\partial f}{\partial t}(x,t).$$

This problem has two parts.

(a) Prove that for all $x \in \mathbb{R}$ and $t > 0$

$$\frac{\partial f}{\partial t}(x,t) = \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(x,t).$$

(b) Prove that for any $\alpha \in \mathbb{R}$

$$\int_{-\infty}^{\infty} \left(\int_{\alpha}^{\infty} g(x,t)dt \right) dx \neq \int_{\alpha}^{\infty} \left(\int_{-\infty}^{\infty} g(x,t)dx \right) dt.$$

(**Hint.** Use part (a) to calculate one of these iterated integrals.)

7. Let m denote Lebesgue measure on $\mathbb{R}$, and let β be a number in $(1, \infty)$. Define the measure ρ on Borel subsets A of $[0, \infty)$ by

$$\rho(A) = \int_A (1 + x^\beta)^{-1} \, dm(x).$$

This problem has three parts.

(a) Prove that ρ is absolutely continuous with respect to m and determine $d\rho/dm$, the Radon-Nikodym derivative of ρ with respect to m.

(b) For which values of $u \in \mathbb{R}$ is $f(x) = x^u$ in $L^1(\rho)$? Justify your answer.

(c) For which values of $u \in \mathbb{R}$ is $f(x) = x^u$ in $L^2(\rho)$? Justify your answer.

8. Let $\mathcal{H}$ be a Hilbert space with norm $\|\cdot\|$ and let $\mathcal{B}(\mathcal{H})$ be the set of bounded linear operators mapping $\mathcal{H}$ into $\mathcal{H}$. For $D \in \mathcal{B}(\mathcal{H})$ define the operator norm

$$\|D\| = \sup\{\|Dx\| : x \in \mathcal{H}, \|x\| = 1\} = \sup\left\{\frac{\|Dx\|}{\|x\|} : x \in \mathcal{H}, x \neq 0\right\}.$$

This problem has two parts.

(a) For D and F in $\mathcal{B}(\mathcal{H})$ and $x \in \mathcal{H}$, define $(DF)x = D(Fx)$. Prove that $\|DF\| \leq \|D\| \cdot \|F\|$.

(b) Prove that every Cauchy sequence $\{D_n, n \in \mathbb{N}\}$ in $\mathcal{B}(\mathcal{H})$ converges in $\mathcal{B}(\mathcal{H})$.