University of Massachusetts
Department of Mathematics and Statistics
Advanced Exam in Geometry
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Do 5 out of the following 8 problems. Indicate clearly which questions you want graded. *Passing standard:* 70% with three problems essentially complete. **Justify all your answers.**

1. Let $f\colon X \to Y$ be a C^∞ map between manifolds. Show that its *graph*

$$\Gamma_f := \{(p, f(p)) \mid p \in X\} \subset X \times Y$$

   is an embedded submanifold of $X \times Y$ diffeomorphic to X.

2. Prove or disprove the following statements:
   (a) If n is even, then every n-form $\alpha \in \Lambda^n(\mathbb{RP}^n)$ vanishes at some point.
   (b) Let $\alpha \in \Lambda^1(S^2)$ and suppose $T^*(\alpha) = \alpha$ for all rotations $T \in SO(3)$. Then $\alpha = 0$.
   (c) If $f\colon \mathbb{R}^n \to \mathbb{R}$ is a submersion, then $N = f^{-1}(17)$ is an orientable manifold.

3. Let M be a manifold, $f\colon M \to \mathbb{R}$ be a smooth map and $p \in M$ a critical point of f.
   (a) Define what is meant by the *Hessian* of f at p. If your definition involves choices, show that the end-result is independent of those choices.
   (b) Show that restricting the quadratic form

$$F(x) = \sum_{j=1}^{n+1} j\, x_j^2\, ; \quad x = (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1}$$

   to unit vectors defines a function $f : \mathbb{RP}^n \to \mathbb{R}$ and find its critical points.
   (c) Compute the Hessian, $\mathrm{Hess}(f)$, at each of the critical points obtained above.

4. Recall that a manifold M is *parallelizable* if its tangent bundle TM is a trivial bundle.
   (a) Characterize parallelizability in terms of vector fields on M.
   (b) Show that S^1 and S^3 are parallelizable.
   (c) Show that if M, N are parallelizable, then so is $M \times N$.
   (d) If $M \times N$ is parallelizable, must both M and N be? Prove or provide a counterexample.

5. Let $Q = \{(x, y) \in \mathbb{R}^2 \colon x > 0, y > 0\}$ with the Lie group structure defined by the product:

$$(x, y) * (x', y') = (xx', yy').$$

(a) Find a basis of left-invariant 1-forms on Q and define a left-invariant metric on Q.

(b) Compute the Gaussian curvature of the metric defined in (a).

(c) Compute the geodesics of Q with respect to the metric defined in (a).

6. Consider the vector fields $V = z\frac{\partial}{\partial x} + x\frac{\partial}{\partial z}$ and $W = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$ on $\mathbb{R}^3$.

   (a) Determine the largest open set $U \subset \mathbb{R}^3$ over which V and W span a 2-dimensional distribution, i.e., a rank 2 subbundle $E \subset TU$ of the tangent bundle TU.

   (b) Find a 1-form α in U such that

   $$E(p) = \{X \in T_p(U) \colon \alpha(p)(X) = 0\} \subset T_p(U)\ ;\ \text{for all } p \in U.$$

   (c) Prove that E is integrable.

   (d) Determine the leaves (maximal integral submanifolds) of E.

7. We identify S^2 with the Riemann sphere $\mathbb{C} \cup \{\infty\}$ and define $A \colon S^2 \to S^2$ by $A(z) = 1/z$ if $z \in \mathbb{C}$, and $A(\infty) = 0$.

   (a) Prove that A is a C^∞ map.

   (b) Show that for all $\omega \in \Lambda^2(S^2)$,

   $$\int_{S^2} A^*(\omega) = \int_{S^2} \omega$$

8. Let M be an even-dimensional manifold. Define what it means for M to be *symplectic* and determine which of the following manifolds are symplectic. (Justify your answers.)

   (a) $S^1 \times S^1$

   (b) $S^2 \times S^2$

   (c) $S^3 \times S^3$.