

DEPARTMENT OF MATHEMATICS AND STATISTICS
UMASS - AMHERST
BASIC EXAM - PROBABILITY
FALL 2006

Work all problems. 60 points are needed to pass at the Master's level and 75 to pass at the Ph.D. level.

1. The logistic function is

$$p = f(x) = \frac{1}{1 + \exp(-x)}, \quad -\infty < x < \infty, 0 < p < 1.$$

- (a) (10 pts) Suppose $X \sim N(0, \sigma^2)$ with pdf $\frac{1}{\sigma\sqrt{2\pi}} \exp\left(\frac{x^2}{-2\sigma^2}\right)$. What is the distribution of $P = f(X)$?
- (b) (10 pts) Find a distribution for X so that $P = f(X)$ has a $\text{Unif}(0,1)$ distribution. For full credit, you should prove that your choice works too.
2. Let Y_1, \dots, Y_n be a random sample from some distribution with $\Pr(Y_i = 1) = \theta, 0 < \theta < 1$. Let $\hat{\theta}_n = n^{-1} \sum_{i=1}^n Y_i$.
- (a) (15 pts) State the central limit theorem in general, and then use it to argue that $n^{1/2}(\hat{\theta}_n - \theta)/\tau$ converges in distribution to a standard normal. Define τ as a function of θ .
- (b) (15 pts) Define a $\hat{\tau}_n$ that converges in probability to τ as n goes to infinity. Explain your answer, and name the results that you use to show convergence in probability.
3. Let $X = 1$ with probability p and $X = 0$ with probability $1 - p$. Let Y be another random variable that can also be either zero or one. Let $\Pr(Y = 1|X = 1) = r$ and $\Pr(Y = 1|X = 0) = s$.
- (a) (10 pts) Find $\Pr(Y = 1)$ and $E(Y)$.
- (b) (10 pts) Find $\text{Var}(Y)$.
- (c) (5 pts) What is the distribution of $Z = X/(Y + 1)$?
4. Let X be a random variable with $E(X) = \lambda$ and $\Pr(X = \lambda) < 1$.
- (a) (10 pts) Does $E(X^{0.5}) = (EX)^{0.5} = \lambda^{0.5}$? Why or why not? If not, give an inequality.
- (b) (15 pts) Suppose $X \sim \text{Exp}(\lambda)$ with pdf $f(x; \lambda) = \frac{1}{\lambda} \exp\left(\frac{-x}{\lambda}\right), x > 0, \lambda > 0$. What is $E(X^{1.5})$? Hint: It may help to recall that the gamma distribution is $g(x, \alpha, \beta) = \frac{x^{\alpha-1} \exp(-x/\beta)}{\Gamma(\alpha)\beta^\alpha}$.