DEPARTMENT OF MATHEMATICS & STATISTICS
BASIC EXAM: NUMERICS
September 2006

Do five of the following problems. All problems carry equal weight.
Passing Level:
Masters: 60% with at least two substantially correct.
PhD: 75% with at least three substantially correct.

1. Newton's method for solving a scalar nonlinear equation $f(x) = 0$ requires computation of the derivative of f at each iteration. Suppose that we instead replace the true derivative with a constant value d, that is, we use the iteration scheme

$$x_{k+1} = x_k - \frac{f(x_k)}{d}.$$

   (a) Under what conditions on the value of d will this scheme be locally convergent.

   (b) What will be the convergence rate, in general, if the order is linear?

   (c) Is there any value for d that would yield quadratic convergence?

2. What is the Cholesky factorization of an $n \times n$ real matrix A, and under what conditions does it exist? Find the Cholesky factorization of

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 5 & 5 \\ 1 & 5 & 14 \end{pmatrix}.$$

3. Consider the two-step method

$$y_{n+1} = \frac{1}{2}(y_n + y_{n-1}) + \frac{h}{4}[4y'_{n+1} - y'_n + 3y'_{n-1}]$$

   with $y'_n = f(x_n, y_n)$. Show that it is second order and find the leading term in the truncation error.

4. Suppose that you have a table of values of the natural logarithm $\ln x$ for positive integer values of x, and you compute $\ln 1.1$ by quadratic interpolation at $x_0 = 10$, $x_1 = 11$ and $x_2 = 12$. Give a good bound on the relative error incurred.

5. Let

$$f(t) = \begin{bmatrix} 0 & \text{if } 0 \le t \le \frac{1}{2} \\ 1 & \text{if } \frac{1}{2} \le t \le 1 \end{bmatrix}$$

   Find the linear least squares approximate p_1 to f on $[0, 1]$. That is, the polynomial of degree 1 for which

$$\int_0^1 [p_1(t) - f(t)]^2 dt = minumum$$

   Use the normal equations.

6. $f(x)$ is a polynomial of degree at most 3. Its value at 9 distinct points is given below:

x	-4	-3	-2	-1	0	1	2	3	4
$f(x)$	-9	0	2	0	-3	-4	0	12	35

Find the **exact** value of $\int_{-4}^{4} f(x)\,dx$. Explain how you are sure that your answer is correct.

7. Let A be a nonsingular $n \times n$ real matrix. Given a vector norm $\|\cdot\|$ on $\mathbb{R}^n$,

   (a) Define the condition number, $\kappa(A)$, and show that $\kappa(A) \geq 1$.

   (b) Show that if A is an orthogonal matrix that $\kappa(A) = 1$ if the Euclidean norm is used.

   (c) Let $Ax = b$ and $(A + E)x = b + c$. Prove that

$$\frac{\|c\|}{\|b\|} \leq \kappa(A)\frac{\|E\|}{\|A\|}.$$