Department of Mathematics and Statistics
University of Massachusetts
ADVANCED EXAM — DIFFERENTIAL EQUATIONS
August 2006

Do five of the following problems. All problems carry equal weight.
Passing level: 75% with at least three substantially complete solutions.

1. Let A and $B(t)$ be $n \times n$ matrices such that

(i) The eigenvalues of A have negative real parts.

(ii) The map $t \longmapsto B(t)$ is continuous and

$$\lim_{t\to\infty} B(t) = 0$$

Show that the zero solution of

$$x' = (A + B(t))x$$

is asymptotically stable.

HINT: Write the solution of the Cauchy problem with $x(t_0) = x_0$ as

$$\begin{aligned} x(t, t_0, x_0) &= e^{(t-T)A}\, x(T, t_0, x_0) \\ &+ \int_T^t e^{(t-s)A}\, B(s)\, x(s, t_0, x_0) ds \end{aligned}$$

for a suitable T.

2. In the positive quadrant $x \geq 0, y \geq 0$ consider the system

$$\begin{aligned} x' &= a - x - \tfrac{4xy}{1+x^2} \\ y' &= bx(1 - \tfrac{y}{1+x^2}) \end{aligned}$$

Where a, b are positive constants.

(i) Construct positively invariant regions to show the existence of solutions of the Cauchy problem for all $t \geq 0$.

(ii) Determine the critical points and their stability properties as a function of (a, b).

(iii) Determine for which values of (a, b) the system has a periodic orbit.

3. Consider the system given, in polar coordinates, by

$$\begin{aligned} r' &= r - r^2 \\ \theta' &= \sin\theta + a \end{aligned}$$

(i) Determine for which values of a the system undergoes bifurcations.

(ii) For all different cases describe qualitatively the critical points (how many? stability? dimension of stable, unstable, center manifolds?) and the periodic orbits (existence? stability?)

(iii) For all different cases describe the ω–limit sets of all solutions.

4. Let $\Omega \subset \mathbf{R}^n$ be a smooth bounded domain.

(a) State the "Trace Theorem" for Ω, pertaining to the Sobolev space $H^1(\Omega)$.

(b) Consider the Poisson equation with "Robin" or "radiation" boundary conditions:

$$\begin{cases} -\Delta u = f & in\ \Omega \\ \frac{\partial u}{\partial N} + \gamma u = 0 & on\ \partial\Omega, \end{cases}$$

wheree $f \in L^2(\Omega)$, $\quad N$ is the <u>outward</u> unit normal on $\partial\Omega$, and γ is a <u>positive</u> constant. Prove the existence <u>and</u> uniqueness of a weak solution to this BVP.

5. Consider the heat equation on the whole line:

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0 \quad (x \in \mathbf{R}^1, t \geq 0).$$

Suppose that the initial data $u(x,0) = \varphi(x)$ are smooth (C^∞) and <u>compactly suppported</u>.

a) Establish that

$$\int_{-\infty}^{+\infty} u(x,t)dx = \int_{-\infty}^{+\infty} \varphi(x)dx = Q \ \ for\ all \ \ t > 0.$$

b) Show that for fixed x and large t,

$$u(x,t) = \frac{Q}{(4\pi t)^{1/2}} e^{-|x|^2/4t} + O(t^{-\frac{3}{2}}).$$

6. Consider the one-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 \quad (x \in \mathbf{R}^1, t \in \mathbf{R})$$

under 2π=periodic boundary conditions; namely,

$$u(x+2\pi,t) = u(x,t), \quad \frac{\partial u}{\partial x}(x+2\pi,t) = \frac{\partial u}{\partial x}(x,t) \ \ for\ all\ t.$$

Let the initial conditions be

$$u(x,0) = \varphi(x), \quad \frac{\partial u}{\partial t}(x,0) = \Psi(x).$$

a) Use a Fourier series to derive an explicit expression for the solution.

b) For $\varphi \in H^m[0,2\pi), \Psi \in H^{m-1}[0,2\pi)$ with any fixed $m \geq 2$, show that the solution u belongs to the space $C^0([0,T]; H^m)$ for each finite T.