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Advanced Analysis Qualifying Examination
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Instructions

1. This exam consists of eight (8) problems all counted equally for a total of 100%.

2. You are encouraged to try to solve every problem; there is no penalty for incorrect answers.

3. In order to pass this exam, it is enough that you solve essentially correctly at least five (5) problems and that you have an overall score of at least 65%.

4. State explicitly all results that you use in your proofs and verify that these results apply.

5. Please write your work and answers clearly in the blank space under each question.

Conventions

1. For a set A, 1_A denotes the indicator function or characteristic function of A.

2. If a measure is not specified, use Lebesgue measure on $\mathbb{R}$. This measure is denoted by m.

3. If a σ-algebra on $\mathbb{R}$ is not specified, use the Borel σ-algebra.

1. For $n \in \mathbb{N}$ consider the following functions on $\mathbb{R}$.

   (a) $f_n = \frac{1}{n} \cdot 1_{(0,n)}$,
   (b) $f_n = 1_{(n,n+1)}$,
   (c) $f_n = n \cdot 1_{[0,1/n]}$,
   (d) $f_n = 1_{[j/2^k,(j+1)/2^k]}$ if $n = 2^k + j, j \in \mathbb{N}, k \in \mathbb{N}, 0 \leq j < 2^k$.

   For each sequence $\{f_n, n \in \mathbb{N}\}$ in (a), (b), (c), (d), determine whether or not $f_n \to 0$ pointwise, $f_n \to 0$ a.e., $f_n \to 0$ uniformly, and $f_n \to 0$ in $L^1(\mathbb{R})$. Explain your answers.

2. Let X be a nonempty set, $\mathcal{P}(X)$ the class of all subsets of X, and μ^* a function mapping $\mathcal{P}(X)$ into $[0, \infty]$.

   (a) What properties must μ^* satisfy if μ^* is an outer measure?

   (b) Let $\mathcal{E} \subset \mathcal{P}(X)$ and $\rho : \mathcal{E} \to [0, \infty]$ be such that $\emptyset \in \mathcal{E}$, $X \in \mathcal{E}$, and $\rho(\emptyset) = 0$. For any $A \subset X$ define

$$\mu^*(A) = \inf\left\{ \sum_{j=1}^{\infty} \rho(E_j) : E_j \in \mathcal{E} \text{ and } A \subset \bigcup_{j=1}^{\infty} E_j \right\}.$$

Prove that μ^* is an outer measure.

3. Let $(X, \mathcal{M}, \mu)$ be a measure space with $0 < \mu(X) < \infty$ and let f be a measurable function on X satisfying $f(x) > 0$ for all $x \in X$.

   (a) Let α be any fixed real number satisfying $0 < \alpha < \mu(X) < \infty$.

   Prove that

$$\inf\left\{ \int_E f d\mu : E \in \mathcal{M}, \mu(E) \geq \alpha \right\} > 0.$$

   (b) Give an example to show that the result in part (a) is false if one drops the hypothesis that $\mu(X) < \infty$.

4. Let $f \in L^1(I\!R)$ be a function mapping $I\!R$ into $I\!R$.

   The Fourier transform $\hat{f}$ of f is defined for $\xi \in I\!R$ by

$$\hat{f}(\xi) = \int_{I\!R} e^{ix\xi} f(x)\, dx\,,$$

   where $i = \sqrt{-1}$.

   (a) Prove that $\hat{f} \in L^\infty(I\!R)$.

   (b) Prove that $\hat{f}$ is a uniformly continuous function.

   (c) Besides assuming that $f \in L^1(I\!R)$, assume that f has compact support.

   Prove that $\hat{f}$ is a C^∞ function.

   Hint. Prove and use the fact that for real numbers x, a, and b, $|e^{ixa} - e^{ixb}| \leq |x|\,|a-b|$.

5. Let $\mathcal{H}$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and associated norm $\|\cdot\|$.

   (a) Let $T : \mathcal{H} \to \mathcal{H}$ be a linear operator. Define the concept that T is a *bounded* linear operator and give the formula for $\|T\|$.

   For the remainder of this problem let $A : \mathcal{H} \to \mathcal{H}$ be a bounded linear operator.

   (b) For fixed $z \in \mathcal{H}$ define $\varphi_z(x) = \langle Ax, z \rangle$ for $x \in \mathcal{H}$. Prove that $\varphi_z : \mathcal{H} \to I\!R$ is a bounded linear functional on $\mathcal{H}$.

   Part (b) implies the following basic fact, which you need not prove: there exists a unique linear operator A^* with the property that

$$\langle Ax, z\rangle = \langle x, A^*z\rangle \text{ for all } x \in \mathcal{H} \text{ and } z \in \mathcal{H}.$$

(c) Prove that $\|A^*\| = \|A\|$.

(d) Prove that $\|A^*A\| = \|A\|^2$.

6. Let s be a fixed positive number. Prove that

$$\int_0^\infty e^{-sx}\,\frac{\sin^2 x}{x}\,dx = \frac{1}{4}\log(1+4s^{-2})$$

by integrating $e^{-sx}\sin(2xy)$ with respect to $x \in (0,\infty)$, $y \in (0,1)$ and with respect to $y \in (0,1)$, $x \in (0,\infty)$. Justify all your steps. (**Hint.** $\cos(2\theta) = 1 - 2\sin^2\theta$.)

7. Let $(X, \mathcal{M}, \mu)$ be a measure space, $\{f_n, n \in I\!\!N\}$ a sequence of measurable functions mapping X into $I\!\!R$, f a measurable function mapping X into $I\!\!R$, and p a real number satisfying $1 \le p < \infty$.

(a) Define the concepts that $f_n \to f$ in measure and that $\|f_n - f\|_p \to 0$ (convergence in $L^p(X)$).

(b) Prove that if $\|f_n - f\|_p \to 0$, then $f_n \to f$ in measure.

(c) Assume that there exists a nonnegative $g \in L^p(X)$ such that $|f_n(x)| \le g(x)$ for all $x \in X$. Prove that if $f_n \to f$ in measure, then $\|f_n - f\|_p \to 0$. In order to do this, use without proof the following formula, valid for any measurable function h on X:

$$\int_X |h|^p\,d\mu = p\int_0^\infty \alpha^{p-1}\,\sigma_h(\alpha)\,d\alpha,$$

where $\sigma_h(\alpha) = \mu(\{x \in X : |h(x)| > \alpha\})$.

8. Define $g(x) = x^{-1/2}$ for $x \in (0,1)$ and $g(x) = 0$ for $x \in I\!\!R \setminus (0,1)$.

(a) Evaluate $\int_{I\!\!R} g(x)dx$ and conclude that $g \in L^1(I\!\!R)$.

(b) Let $\{r_k, k \in I\!\!N\}$ be the set of rational numbers in $I\!\!R$ and let $\{a_k, k \in I\!\!R\}$ be any sequence of real numbers satisfying $a_k > 0$ and $\sum_{k\in I\!\!N} a_k < \infty$. In terms of these quantities define for $x \in I\!\!R$

$$f(x) = \sum_{k=1}^\infty a_k\,g(x - r_k).$$

(i) Prove that $f \in L^1(I\!\!R)$.

(ii) Prove that $f \notin L^2([\alpha,\beta])$ for any real numbers α and β satisfying $-\infty < \alpha < \beta < \infty$.