DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS
BASIC EXAM - NUMERICS
August, 2004

Do five of the following problems. All problems carry equal weight.
Passing level:
Masters: 60% with at least two substantially correct.
Ph.D.: 75% with at least three substantially correct.

1. The equation $x^2 - a = 0$ (for the square root $\alpha = \sqrt{a}$) can be written equivalently in the form $x = \phi(x)$ in many different ways, for example,

   (a) $\phi(x) = \frac{1}{2}(x + \frac{a}{x})$

   (b) $\phi(x) = \frac{a}{x}$

   (c) $\phi(x) = 2x - \frac{a}{x}$

   Discuss the convergence (or nonconvergence) behavior of the iteration $x_{n+1} = \phi(x_n)$, $n = 0, 1, 2, ...$, for each of these three iteration functions. In case of convergence, determine the order of convergence.

2. (a) Derive an approximation formula for the second derivative,

   $$f''(x) = A\ f(x) + B\ f(x+h) + C\ f(x+2\,h) \equiv S(x,h)$$

   which is as accurate as possible.

   (b) Derive an expression for the error of this approximation.

   (c) Use Richardson's extrapolation with $S(x,h)$ and $S(x,h/2)$ to obtain a better approximation. What can you say about the error?

3. Find the coefficients a_i and nodes x_1 and x_2 so that the quadrature formula

$$\int_{-1}^{1} f(x)\,dx \approx a_0\, f(-1) + a_1\, f(x_1) + a_2\, f(x_2) + a_3\, f(1)$$

has the highest possible degree of precision. What is the degree of precision?

4. Find a cubic spline that interpolates the data

$$f(0) = 0, \quad f(1/3) = 1/2, \quad f(1) = 1,$$

and satisfies the natural boundary conditions.

5. Consider the linear system

$$\begin{pmatrix} 5 & 2 & 2 \\ 2 & 5 & 2 \\ 2 & 2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}.$$

   (a) Write down the Gauss-Siedel method for solving the system.

   (b) Give the iteration matrix of this iteration and compute its spectral radius.

6. Derive a multi-step method

$$y_{n+1} = y_n + h\,(A\, f_n + B\, f_{n-1} + C\, f_{n-2})$$

for the ODE $y' = f(t, y)$ which is third-order accurate, and compute the truncation error.

7. (a) Calculate the condition number of A, $cond(A) = \|A\|_p \|A^{-1}\|_p$, where

$$A = \begin{pmatrix} 100 & 99 \\ 99 & 98 \end{pmatrix}$$

for each of $p = 1$ and $p = \infty$.

(b) Compare these values with

$$\frac{\max|\lambda_i|}{\min|\lambda_i|}$$

where λ_i is an eigenvalue of A.