DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS
MASTER'S OPTION EXAM-APPLIED MATHEMATICS
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Do five of the following problems. All problems carry equal weight.
Passing level: 60% with at least two substantially correct.

1. Consider the ordinary differential equation

$$\frac{dx}{dt} = x^2 - 2x - 3$$

   (a) What are the constant solutions?

   (b) Sketch the solutions of the ODE with initial data $x(0) = -2, 0, 4$ and find $\lim_{t \to \pm\infty} x(t)$ for each solution. Do not solve the equation!

2. Consider the predator-prey system

$$\frac{dx}{dt} = (2 + x)(y - x)$$

$$\frac{dy}{dt} = (4 - x)(y + x)$$

   (a) Determine all critical points of the system.

   (b) Find the corresponding linear system near each critical point.

   (c) Discuss the stability of the solution near each critical point.

3. (a) Give a physical interpretation of the equation

$$u_t + xu_x = 0$$

(b) Draw the characteristics and solve the above equation with initial data $u(x, 0) = sin(x)$.

4. Find all solutions to

$$u_{xx} + u_{yy} = 0$$

in the rectangle $0 < x < a$, $0 < y < b$, with the following boundary conditions

$$u_x = -a \text{ on } x = 0, \ \ u_x = 0 \text{ on } x = a.$$

$$u_y = b \text{ on } y = 0, \ \ u_y = 0 \text{ on } y = b.$$

(Hint: Try $u(x, y) = X(x) + Y(y)$ not $u(x, y) = X(x)Y(y)$.)

5. Solve the eigenvalue problem

$$x^2 u'' + 3xu' + \lambda u = 0$$

for $1 < x < e$, with $u(1) = u(e) = 0$. (Hint: Look for solutions of the form $u = x^m$.) For what values of $\lambda > 1$ does this equation have a nontrivial solution?

6. Consider the equation

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} - \alpha u,$$

which corresponds to a one-dimensional rod, either with heat loss through the ends with outside temperature 0 degrees ($\alpha > 0$, $k > 0$), or with insulated ends and with a heat source proportional to the temperature ($\alpha < 0$). Suppose that the boundary conditions are

$$u(0, t) = 0 \text{ and } u(L, t) = 0.$$

(a) What are the possible equilibrium temperature distributions if $\alpha > 0$ and using the boundary conditions above?

(b) Solve the time-dependent problem $[u(x,0) = f(x)]$ for $\alpha > 0$. Analyze the temperature for large time($t \to \infty$) and compare with part (a).

7. In 1-dimensional gas dynamics, the trajectory of a particle approaching a vacuum is given by
$$\frac{dX}{dt} = c,$$
where $c \geq 0$ is the sound speed, and $\frac{dc}{dt} < 0$. The trajectory can also be given by
$$X(c) = -c\,t(c) + H(c),$$
where $H(c)$ is regarded as a known decreasing function. Combining these we get the equation
$$\frac{d}{dc}(c\,t) + c\,\frac{dt}{dc} = H'(c),$$
where $H'(c) \leq 0$ is known. Solve to get an integral expression for $t(c)$, with data $t(1) = 1$, and $c < 1$. Show that $t(c) \geq \frac{1}{\sqrt{c}}$ and use this to conclude that the vacuum state $(c = 0)$ *cannot* be reached in finite time. [Hint: find an integrating factor.]