Department of Mathematics and Statistics
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Basic Exam: Linear Algebra/Advanced Calculus

August 25, 2003

Do 7 of the following 9 problems. Indicate clearly which problems should be graded.

Passing Standard: For Master's level, 60% with three questions essentially complete (including at least one from each part). For Ph. D. level, 75% with two questions from each part essentially complete.

Part I Linear Algebra

1. Let A be an $m \times n$ matrix over $\mathbb{R}$. If $A^t Ax = 0$ for some $x \in \mathbb{R}^n$, show that $Ax = 0$. Use this to show that if the columns of A are linearly independent, then $A^t A$ is invertible. [Hint: Consider $\langle Ax, Ax \rangle$ where $\langle x, y \rangle$ is the usual inner product.]

2. Let $T : V \to V$ be a linear operator on a finite dimensional vector space. Prove that there is an integer m for which $(\operatorname{Ker} T^m) \cap (\operatorname{Im} T^m) = 0$.

3. Suppose a and b are nonzero real numbers. Consider the matrix

$$A = \begin{pmatrix} 1 & a & b \\ a & a^2 & ab \\ b & ab & b^2 \end{pmatrix}.$$

   (a) Determine the nullity of A (the dimension of $\operatorname{Ker} A$).

   (b) Find two orthogonal eigenvectors for A.

   (c) Must $\mathbb{R}^3$ have an orthogonal basis consisting of eigenvectors for A?

4. Let $T : V \to V$ be a linear operator on a finite dimensional vector space, with characteristic polynomial $f(x)$.

   (a) Suppose T has two linearly independent eigenvectors with the same eigenvalue λ. Must λ be a multiple root of $f(x)$? Give proof or counterexample.

   (b) Suppose μ is a multiple root of $f(x)$. Must T have two linearly independent eigenvectors with eigenvalue μ? Give proof or counterexample.

Part II Advanced Calculus

1. Let $f : S \to \mathbb{R}$ be *uniformly continuous* on a subset S of $\mathbb{R}$.

   (a) If (x_n) is a Cauchy sequence in S, prove that $(f(x_n))$ is a Cauchy sequence in $\mathbb{R}$.

   (b) If S is bounded, prove that f is bounded.

2. Define the natural logarithm function for $x > 0$ by

$$\ln(x) := \int_1^x \frac{1}{t}\, dt$$

   (a) Prove that ln is differentiable everywhere and hence continuous.

   (b) Prove that $\ln(ab) = \ln(a) + \ln(b)$ for all $a, b > 0$. [Use a change of variable.]

   (c) Noting that $\ln(1) = 0$ and $\ln'(1) = 1$, use the definition of the derivative to prove that $\ln(e) = 1$, where

$$e := \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$$

3. Suppose $f : [0,1] \to [0,1]$ is continuous.

   (a) Prove (using only the methods of calculus) that $f(x) = x$ for some $x \in [0,1]$.

   (b) Starting with any $c \in [0,1]$, define a sequence $\{x_n\}$ inductively by $x_1 = c$ and $x_{n+1} = f(x_n)$. Suppose $\{x_n\}$ converges to a point x. Prove that $f(x) = x$.

4. Find a local maximum value of $f(x,y,z) = xy^2z^2$ on the plane $x + y + z = 12$.

5. Let C be the triangular boundary of the plane $6x + 3y + 2z = 6$ in the first octant. Compute $I = \oint\limits_C F \cdot \vec{ds}$ for the vector field F given by $F(x,y,z) = (yz, -xz, xy)$. [Hint: Use Stokes' Theorem.]