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ADVANCED EXAM — DIFFERENTIAL EQUATIONS
AUGUST 30, 2002

Do five of the following problems. All problems carry equal weight.
Passing level: 75% with at least three substantially complete solutions.

1. a) Find all eigenvalues of the linearization of the the system

$$\begin{aligned} u_1' &= -u_1 + u_2 - u_1 u_3^2 \\ u_2' &= u_1 - 2u_2 + u_3 \\ u_3' &= u_2 - u_3 - u_1^2 u_3 \end{aligned}$$

about the rest point at the origin and determine its stable and unstable manifolds. What geometric conclusion can you draw from this information regarding the behavior of solutions of the nonlinear system in a neigborhood of the origin?

b) Prove that the rest point at the origin is stable.
(hint: consider the function $E(u_1, u_2, u_3) = u_1^2 + u_2^2 + u_3^2$).

2. a) Give a complete proof of the following:
If $f(X)$ is a C^1 mapping from a bounded domain G in R^n into R^n, and G is positively invariant for the flow determined by $X' = f(X)$, then given $T > 0$ and $\varepsilon > 0$ there is a constant $K > 0$ depending only on f and G such that

$$||X(t) - \tilde{X}(t)|| \leq e^{Kt} ||X(0) - \tilde{X}(0)||$$

for all $t \geq 0$, whenever $X(0), \tilde{X}(0)$ lie in G, where the indicated norm is the Euclidean norm.

b) Show that the square $G = \{0 \leq x, y \leq 1\}$ is positively invariant for the system

$$\begin{aligned} x' &= x - x^2 - 3xy \\ y' &= y - 2xy - y^2. \end{aligned}$$

c) Find a value $\varepsilon > 0$ such that $||X(t) - \tilde{X}(t)|| \leq .01$ for any t, $0 \leq t \leq 4$ for the system in b) (with $X = (x, y)$), whenever $||X(0) - \tilde{X}(0)|| < \varepsilon$.

3. a) Find the solution $u(x,t)$ of the initial value problem

$$u_t + x^3\, u_x = 0, \quad u(x,0) = u_0(x)$$

with $x \in R$.

b) For u_0 continuous and x fixed, find

$$\lim_{t\to\infty} u(x,t).$$

4. Consider the biharmonic equation

$$\Delta^2 u = f \quad \text{in} \quad \Omega, \quad \text{with data}$$
$$u = 0, \quad \frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \partial\Omega,$$

where Ω is a regular bounded open subset of the plane R^2, and $f \in L^2$.

a) Define a weak solution for this problem. Be sure to explain your definition.

b) Show that any weak solution which is also a C^4 function is a classical solution.

5. Consider the heat equation in the domain $\Omega \times (0,\infty)$, and satisfying the Neumann condition

$$\frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \partial\Omega \times (0,\infty).$$

a) Write down a series expansion of the solution to the initial value problem in terms of the eigenfunctions ϕ_n of the Laplacian on the domain Ω.

b) For Ω the unit disk and for initial data

$$u(x,0) = u_0(|x|),$$

give the solution explicitly by calculating its eigenfunction expansion.

6. a) Consider the system

$$\begin{aligned} x' &= y \\ y' &= f(x) \end{aligned}$$

where $f(x)$ is a smooth function with roots at $a < 0 < b < 1$, with $f'(0), f'(1) < 0$ and $f'(a), f'(b) > 0$. Suppose that the (absolute) area under f on $0 \leq x \leq b$ is less than the area under f on $b \leq x \leq 1$. Sketch the phase plane of all solutions of the above system.

b) Show that there must be a connecting orbit running from $(0,0)$ to $(1,0)$ for the system

$$\begin{aligned} x' &= y \\ y' &= f(x) - \theta y \end{aligned}$$

for some value of the parameter $\theta < 0$ along which both $x(t)$ and $y(t)$ are positive for all t.

7. Solve the wave equation in the domain $(0,\infty) \times (0,\infty)$ with initial conditions

$$u(x,0) = f(x), \quad \frac{\partial u}{\partial t}(x,0) = g(x) \quad \text{for} \quad x > 0,$$

and boundary condition

$$\frac{\partial u}{\partial x}(0,t) = 0 \quad \text{for} \quad t > 0.$$