DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS
BASIC EXAM – PROBABILITY
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Work all problems. 60 points are needed to pass at the Master's level and 75 to pass at the Ph.D. level.

1. (17 points) In a children's game a six-sided die is rolled until all six faces have come up. Let T be the number of rolls it takes for this to happen. For example, for the sequence:

$$\underbrace{11243665\ldots}_{T=8}$$

   (a) Compute the expectation of T.

   (b) Compute the variance of T.

2. (16 points) Let X have a Poisson distribution with parameter λ,

$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}$$

   for $x = 0, 1, \ldots$.

   (a) Derive the moment generating function of X.

   (b) Suppose $X_1, \ldots, X_N$ are independent random variables with $X_i \sim \text{Poisson}(\lambda_i)$ What is the moment generating function of $Y = \sum_{i=1}^{N} X_i$?

   (c) What is the distribution of Y? You must justify your answer.

3. (17 points) An advertising agency sends out periodic mailings to two clients, *American Buzz Saw Inc.*, and *Bailey's Fine Fabrics*. For each mailing, the president of the agency sends a letter by messenger to her secretary; however, the letter gets lost with probability 1/4. If the secretary receives the letter, he sends a copy to each of the two clients. The letter to ABS has probability 4/5 of being received while the one to BFF has probability 1/6 of being lost. The letters to the clients, if sent, are received or lost independently of each other. Let S be the event "secretary receives letter," A be the event "ABS receives letter," and B be the event "BFF receives letter." Thus, for example $P(A|S) = 4/5$. If both clients receive their letters, the mailing is deemed successful.

   (a) Find $P(B|S^c)$.

   (b) Find $P(A^c)$ and $P(S|A^c)$.

   (c) Determine whether or not the events A and B are independent.

   (d) Suppose there are n mailings and that the behavior of the system is independent for each item. Find the mean and variance of the number of successful mailings.

4. (17 points) In this problem you may use any properties of the standard normal density that you need. Let

$$f(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\{-\frac{x^2 - 2\rho xy + y^2}{2(1-\rho^2)}\}$$

for $x, y \in \Re$ where $-1 < \rho < 1$.

   (a) Verify that $f(x,y)$ is a probability density function in the xy-plane.
   (b) Let (X,Y) be the random vector whose joint density is $f(x,y)$. Compute $E(X)$. (show the computation)
   (c) Find the marginal distribution of Y.
   (d) Find the conditional density of X given $Y = y$. Which distribution is it?

5. (17 points) Let X_1 and X_2 be independent, identically distributed random variables having common pdf

$$f(x) = \begin{cases} e^{-x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

Let

$$\begin{aligned} Y_1 &= X_1 + X_2 \\ Y_2 &= X_1 - X_2 \end{aligned}$$

   (a) Find the joint pdf of Y_1 and Y_2.
   (b) Find the marginal pdf of Y_1. What is the distribution of Y_1?
   (c) Find the conditional pdf of Y_2, given $Y_1 = y_1$, for some fixed y_1.

6. (16 points) Joe and Chris go the the Blue Wall every day and flip a coin to decide who will buy coffee for the other at a price of \$1. Joe says they needn't keep track of the history since things will "average out" and not matter in the long run. Let T_n be Joe's "debt" to Chris after n trips to the BW (T_n could be negative). For all of the following questions you must justify your answers. Let $c > 0$.

   (a) What is $\lim_{n\to\infty} P(|T_n/n| < c)$?
   (b) Give a function $s(n)$ so that $\alpha = \lim_{n\to\infty} P(|T_n/s(n)| < c)$ satisfies $0 < \alpha < 1$. What is α? (your answer needn't be a number, but should be computable from a table).
   (c) What is $\lim_{n\to\infty} P(|T_n| < c)$?
   (d) Do you agree with Joe's assertion?