DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS
ADVANCED EXAM – ANALYSIS
29 AUGUST 2001

Instructions: The exam consists of eight problems, all counted equally. You are encouraged to try to solve every problem; there is no penalty for incorrect answers. In order to pass the exam, it is enough to solve essentially correctly at least five problems and to have an overall score of at least 65%. State explicitly all results that you use in your proofs and verify that these results apply.

1. (a) If $E \subset \mathbb{R}$, define $m^*(E)$, the Lebesgue outer measure of E.

   (b) Prove that if $E \subset F \subset \mathbb{R}$, then $m^*(E) \leq m^*(F)$.

   (c) Let A be a subset of $\mathbb{R}$ satisfying $m^*(A) = 0$. Prove directly that A is Lebesgue-measurable.

2. Let $(X, \mathcal{M}, \mu)$ be a measure space.

   (a) State Fatou's Lemma.

   (b) State the Dominated Convergence Theorem.

   (c) Assuming Fatou's Lemma, prove the Dominated Convergence Theorem.

3. Let $(X, \mathcal{M}, \mu)$ be a measure space.

   (a) Assume $f : X \to [0, \infty]$ is measurable. Show that if $\int_X f d\mu = 0$, then $f = 0 \ \ \mu - a.e.$

   (b) Assume $f : X \to \mathbb{R}$ is μ-integrable. Show that if $\int_A f d\mu = 0$ for every $A \in \mathcal{M}$, then $f = 0 \ \ \mu - a.e.$

4. Let m denote Lebesgue measure on $[0, 1]$. Let $\{g_n, n \in \mathbb{N}\}$ be a sequence of measurable functions on $([0, 1], \mathcal{B}_{[0,1]}, m)$ and g a measurable function on $([0, 1], \mathcal{B}_{[0,1]}, m)$. We say that g_n converges to g in distribution, and write $g_n \xrightarrow{d} g$, if for every bounded continuous function f

$$\int_{[0,1]} (f \circ g_n) dm \to \int_{[0,1]} (f \circ g) dm.$$

   (a) Prove that if $g_n \to g \ \ a.e.$, then $g_n \xrightarrow{d} g$.

(b) Assume that there exists $M < \infty$ such that $|g_n| \leq M$ for all $n \in I\!N$ and $|g| \leq M$. Prove that if $g_n \to g$ in measure, then $g_n \stackrel{d}{\to} g$. (**Hint.** A bounded continuous function f is uniformly continuous on $[-M, M]$).

5. Let F be an absolutely continuous function on $[a, b]$.

(a) In what sense does the derivative of F exist?

(b) Define

$$\overline{F} = \frac{1}{b-a} \int_a^b F(y)dy.$$

Show there is a constant C independent of F such that

$$\int_a^b |F(x) - \overline{F}|dx \leq C \int_a^b |F'(x)|dx.$$

6. Let $\mathcal{H}$ be a Hilbert space with inner product $< \cdot, \cdot >$ and norm $\| \cdot \|$ and let $\{v_n, n \in I\!N\}$ be an orthonormal basis of $\mathcal{H}$. For $k \in I\!N$ define a linear operator P_k mapping $\mathcal{H}$ into $\mathcal{H}$ by $P_k x = \sum_{n=1}^{k} <x, v_n> v_n$.

(a) Calculate $\| P_k \|$.

(b) Prove that for any $x \in \mathcal{H}$, $P_k x \to x$ in $\mathcal{H}$ as $k \to \infty$.

(c) Is the convergence in part (b) uniform over $x \in \mathcal{H}$? Either prove or give a counterexample.

7. Let $1 \leq p < \infty$. For $f \in L^p(\mathbf{T}), k \in Z\!\!\!Z$ and $N \in I\!N$ define

$$\hat{f}(k) = \int_{\mathbf{T}} f(x) e^{-2\pi i k x} dx$$

on the 1-dimensional torus $\mathbf{T} = [0, 1)$ and define

$$S_N(f)(x) = \sum_{k=-N}^{N} \hat{f}(k) e^{2\pi i k x}.$$

(a) Assume that for every $1 \leq p < \infty$ there exists a constant $C_p > 0$ independent of $N \in I\!N$ such that

$$\| S_N(f) \|_{L^p(\mathbf{T})} \leq C_p \| f \|_{L^p(\mathbf{T})}$$

for every f in $L^P(\mathbf{T})$. Prove that as $N \to \infty$,

$$S_N(f) \longrightarrow f \text{ in } L^P(\mathbf{T}) \text{ for every } f \in L^P(\mathbf{T}).$$

(b) Explain briefly why for $p = 2$

$$\| S_N(f) \|_{L^2(\mathbf{T})} \leq \| f \|_{L^2(\mathbf{T})}$$

holds for every $N \in I\!N$ and every $f \in L^2(\mathbf{T})$.

8. Let $h \in I\!R$ and let $Z_h(f) = f(x - h)$, where f maps $I\!R$ into $I\!R$.

(a) Prove that if $1 \leq p < \infty$ and $f \in L^P(I\!R)$, then

$$\| Z_h(f) - f \|_{L^p(I\!R)} \longrightarrow 0$$

as $h \longrightarrow 0$. (**Hint.** Use the fact that $C_c^\infty(I\!R)$ is dense in $L^p(I\!R)$ for $1 \leq p < \infty$.)

(b) Prove that when $p = \infty$ the conclusion in part (a) is false.