DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS, AMHERST

ADVANCED EXAM — ALGEBRA

AUGUST 27, 2001

Passing Standard: It is sufficient to do FIVE problems correctly, including at least ONE FROM EACH of the four parts.

All rings are commutative with 1, and every ring homomorphism takes 1 to 1.

Part I.

1. Show that there is no simple group of order 36. (Hint: You may quote the Sylow theorems, but otherwise provide all details.)

2. How many homomorphisms are there from the group $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ to the dihedral group of order 8? (Hint: you can use the fact that the dihedral group of order 8 has exactly two subgroups isomorphic to $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$.)

Part II.

1. (a) Prove that a finite integral domain must be a field.
   (b) Determine all prime ideals and maximal ideals of the ring $\mathbf{Z}/n\mathbf{Z}$, for $n > 1$.

2. Determine all ring automorphisms of the polynomial ring $\mathbf{Z}[x]$.

Part III.

1. Determine an integer m such that $(\mathbf{Z}/10\mathbf{Z} \oplus \mathbf{Z}/21\mathbf{Z}) \otimes_{\mathbf{Z}} (\mathbf{Z}/4\mathbf{Z} \oplus \mathbf{Z}/9\mathbf{Z}) \simeq \mathbf{Z}/m\mathbf{Z}$ as $\mathbf{Z}$-modules.

2. Let M be the $n \times n$ matrix over a field K of characteristic zero such that every entry of M is 1.
   (a) Find the characteristic polynomial of M.
   (b) Determine the Jordan canonical form of M over K.

Part IV.

1. Let F_1, F_2 be two intermediate subfields of a Galois extension K/k. Let $H_1 = \mathrm{Gal}(K/F_1)$ and $H_2 = \mathrm{Gal}(K/F_2)$. Show that H_1 and H_2 are conjugate (as subgroups) in $\mathrm{Gal}(K/k)$ if and only if there exists an automorphism $\tau \in \mathrm{Gal}(K/k)$ such that $\tau(F_1) = F_2$.

2. (a) Compute the minimal polynomial of $\sqrt{2} + \sqrt{-2}$ over $\mathbf{Q}$.
   (b) Determine the Galois group of the Galois closure of the extension $\mathbf{Q}(\sqrt{2}+\sqrt{-2})/\mathbf{Q}$.

Note: Although it is NOT essential for this problem, you may find it convenient to use the fact that the resolvent cubic of $x^4+bx^3+cx^2+dx+e$ is $x^3-cx^2+(bd-4e)x-b^2e+4ce-d^2$.