Analysis Qualifying Examination

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This exam consists of eight equally weighted problems (ten points each): a passing grade is 65% (52/80), including at least five “essentially correct” problems ($\approx 7.5/10$).

Clearly show your work, explicitly stating or naming results that you use; justify use of named theorems by verifying necessary conditions.

Please work legibly and clearly label each page/file of your exam with your name.

Unless otherwise stated, the measure in every problem is the Lebesgue measure. Unless otherwise specified, the underlying space is $\mathbb{R}^d$. 
1. Let $f$ be a non-negative, Lebesgue integrable function on $\mathbb{R}$. Denote by $\mathcal{B}(\mathbb{R})$ the Borel $\sigma$-algebra of $\mathbb{R}$ and by $m$ the Lebesgue measure on $\mathbb{R}$. For $A \in \mathcal{B}(\mathbb{R})$ define

$$
\mu(A) = \int_A f \, dm.
$$

(a) Prove that $\mu$ is a measure and explain why it is finite.

(b) Prove that for any $\epsilon > 0$ there exists $\delta > 0$ such that for every $E \in \mathcal{B}(\mathbb{R})$ satisfying $m(E) < \delta$ we have

$$
\int_E f \, dm < \epsilon.
$$
2. Let \( X \) be a Banach space with norm \( \| \cdot \| \) and let \( L(X, X) \) be the space of all bounded, linear operators mapping \( X \) into \( X \).

(a) For \( U \in L(X, X) \) give the definition of \( \| U \| \) (the operator norm).

(b) Assume that \( U \in L(X, X) \) satisfied \( \| I - U \| < 1 \), where \( I \) is the identity operator. Prove that \( U \) is invertible and that \( \sum_{n=0}^{\infty} (I - U)^n \) converges in \( L(X, X) \) to \( U^{-1} \).

(c) Assume that \( U \in L(X, X) \) is invertible and that \( W \in L(X, X) \) satisfies \( \| W - U \| < \| U^{-1} \|^{-1} \). Prove that \( W \) is invertible.
3. Let $P$ be the orthogonal projection associated with a closed subspace $S$ in a Hilbert space $H$, that is,

$$P(f) = f \quad \text{if } f \in S \quad \text{and} \quad P(f) = 0 \quad \text{if } f \in S^\perp.$$ 

(a) Prove that $P^2 = P$ and $P^* = P$.

(b) Conversely, if $Q$ is any bounded operator on $H$ satisfying $Q^2 = Q$ and $Q^* = Q$, prove that $Q$ is the orthogonal projection for some closed subspace of $H$.

(c) Give an example (you can choose $H$) where $Q^2 = Q$, $Q^* \neq Q$, and $Q$ is not an orthogonal projection.
4. Let \( X = [0, 1] \) endowed with the Lebesgue measure and let \( f(x, y) = \frac{x^2 - y^2}{(x^2 + y^2)^2} \) if \((x, y) \neq (0, 0)\) and \( f(0, 0) = 0 \).

(a) Use the trigonometric substitution \( x = \tan \theta \) to prove that

\[
\int \frac{x^2 - a^2}{(x^2 + a^2)^2} dx = -\frac{x}{x^2 + a^2}.
\]

Hint: Recall the identities \( \sin^2 \theta = \frac{1}{2}(1 - \cos(2\theta)) \), \( \cos^2 \theta = \frac{1}{2}(1 + \cos(2\theta)) \), and \( \sin(2\theta) = 2\sin \theta \cos \theta \).

(b) Use part (a) to compute the integrals

\[
\int_0^1 \left( \int_0^1 f(x, y) dm(x) \right) dm(y) \quad \text{and} \quad \int_0^1 \left( \int_0^1 f(x, y) dm(y) \right) dm(x),
\]

and show that they are not equal.

(c) In the example of (b) which hypotheses of the Fubini-Tonelli Theorem are violated?
5. Let \((X, M, \mu)\) be a measure space. Let \(\{f_n, n \in \mathbb{N}\}\) be a sequence of integrable functions that converges in measure to another integrable function \(f \in L^1(\mu)\). Define \(g(x) = \sup_{n \in \mathbb{N}} |f_n(x)|\) for \(x \in X\) and assume that \(g\) is integrable. Recall that \(f_n\) is said to converge to \(f\) in measure if for every \(\epsilon > 0\) we have \(\lim_{n \to \infty} \mu(\{|f_n - f| \geq \epsilon\}) = 0\). (You may use the fact that this implies the existence of a subsequence that converges a.e. to \(f\).)

(a) Prove that \(\lim_{n \to \infty} \int_X f_n d\mu = \int_X f d\mu\). Hint: Consider \(g \pm f_n\) and Fatou (justify use of Fatou carefully).

(b) Prove that \(f_n\) converges to \(f\) in \(L^1\).
6. In this problem $k$ denotes a fixed integer strictly larger than $\frac{d}{2}$ and $\mathcal{S}$ denotes the space of Schwartz functions.

(a) Prove that there is a constant $C > 0$ such that for all $f \in \mathcal{S}$

$$\|f\|_{L^\infty} \leq C \sum_{|\alpha| \leq k} \|\partial^\alpha f\|_{L^2}.$$ 

Hint: Use the Fourier transform.

(b) Prove that there is a constant $C > 0$ such that for all $f \in \mathcal{S}$

$$\|f\|_{L^\infty} \leq C \|f\|_{L^2}^{\frac{2k-d}{2k}} \left( \sum_{|\alpha| = k} \|\partial^\alpha f\|_{L^2} \right)^{\frac{d}{2k}}.$$ 

Hint: Use part (a) and a scaling argument.
7. Give examples, with full justification, of sequences of functions (for your choice of measure space in each case) such that

(a) $f_n \to f$ uniformly, but $f_n$ does not converge to $f$ in $L^1$ (with $f_n$ and $f$ integrable).

(b) $f_n \to f$ a.e., but $f_n$ does not converge to $f$ in measure. Recall: we say that $f_n$ converges to $f$ in measure if for all $\epsilon > 0$, $\mu(\{|f_n - f| \geq \epsilon\}) \to 0$ as $n \to \infty$.

(c) $f_n \to f$ in $L^1$, but $f_n(x)$ does not converge to $f(x)$ for any $x$. Hint: Consider $f_n = \chi_{I_n}$ to be the characteristic functions of appropriate intervals $I_n \subset \mathbb{R}$.

(d) $f_n \to f$ in measure, but $f_n$ does not converge to $f$ in $L^1$ (with $f_n$ and $f$ integrable).

(e) $f_n \to f$ weakly in $L^2$ but not in $L^2$. Recall that $f_n$ is said to converge to $f$ weakly in $L^2$ if $\int f_n g \to \int f g$ as $n \to \infty$ for all $g \in L^2$. 

8. (a) If \( f : [0, 1] \rightarrow \mathbb{R} \) is bounded and increasing, prove that it is of bounded variation.

(b) Prove that \( f(x) : [0, 1] \rightarrow \mathbb{R} \) defined by \( f(x) = x \sin(x^{-1}) \) for \( x \neq 0 \) and \( f(0) = 0 \), is not of bounded variation.

(c) More generally, if \( a, b > 0 \), let

\[
 f(x) = \begin{cases} 
 x^a \sin(x^{-b}) & \text{for } 0 < x \leq 1, \\
 0 & \text{if } x = 0.
\end{cases}
\]

Prove that \( f \) is of bounded variation on \([0, 1]\) if and only if \( a > b \). Hint: one approach is to find the maxima and minima of \( f \) and directly use the definition by refining a given partition.