Basic Exam: Advanced Calculus & Linear Algebra

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Instructions: Do 5 of the 6 calculus problems and all 5 of the linear algebra problems. Show your work. The passing standards are:

- Master's level: 60% with three questions essentially, complete (including one question from each part);
- Ph.D. level: 75% with two questions from each part essentially complete.

Calculus

1. Consider the region of the paraboloid $z = x^2 + y^2$ below the plane $z = 1$.
   a) Write the integral describing its surface area.
   b) Compute that area. Explain your reasoning and computations.

2. Your carpenter has had a mystic vision: from now on, every rectangular table he builds must satisfy the sacred cubic equation
$$w^3 + \ell^3 = 16,$$
where w is the table's width and ℓ is its length. On the other hand, due to supply issues from COVID, the wood he uses is extremely expensive: the cost per area A scales exponentially so $c(A) = Be^A$, where the coefficient $B > 0$ depends on the choice of currency. Find the maximum cost for this carpenter to build a table and also describe the range of costs to build a table. Explain your reasoning and computations.

3. Sitting on the bus, you meet a friendly smooth function $f : (0, 20) \to \mathbb{R}$. It's a little hard to understand, except very close to the input $t = 10$, where you caught that $f(10) = 1$, $f'(10) = 1/2$, and $f''(10) = -2$. The function won't tell you its third derivative, but it does divulge that $f'''(t)$ always lies between -3 and 3.
   a) Estimate $f(8)$ based on what you know and explain your reasoning.
   b) Describe and justify the error bars on your estimate.

4. Consider the integral equation
$$a + \int_1^x \frac{f(t)}{t} \mathrm{d}t = \operatorname{arcsec}(x)$$
for $x > 1$. Find the constant a and the function f, and justify your answer.

5. A team of exobiologists are attempting to trap a dangerous, alien gas-creature inside the unit cube $C = [0, 1]^3 \subset \mathbb{R}^3$, . At this very moment it is flowing along the field $\mathbf{F}(x, y, z) = (2x, 3y, z^2)$, where (x, y, z) denotes a point inside the cube. What is the total flux of this miasma out of the cube's surface? Explain your reasoning and computations.

6. Let $f(x)$ be the power series
$$\sum_{n=1}^{\infty} \frac{x^{n!}}{n+1}.$$
Solve the differential equation $xF' = f$ and explain your reasoning. (You do not need to discuss domains of convergence.)

Linear Algebra

7. Let $\mathbb{P}_3(\mathbb{R})$ denote the real vector space of polynomials with real coefficients of degree at most 3. Let $T\colon \mathbb{P}_3(\mathbb{R}) \to \mathbb{C}$ the linear map of $\mathbb{R}$-vector spaces defined by

$$p(x) \mapsto 3 \cdot p(0) - i \cdot p''(1) + p'(i).$$

Compute a basis for the kernel of T.

8. Let A be the (3×2)-matrix given by

$$A = \begin{pmatrix} 1 & 2 \\ -2 & 1 \\ -1 & 0 \end{pmatrix}.$$

a) Find a vector of length 5 in the orthogonal complement of the column space of A.

b) Find an orthonormal basis of the orthogonal complement of the null space of A.

9. For $k \geq 0$, the (3×3)-vector $\mathbf{x}_k$ is defined by

$$\mathbf{x}_0 = \begin{pmatrix} 5 \\ 1 \\ 7 \end{pmatrix} \quad \text{and} \quad \mathbf{x}_{k+1} = A\mathbf{x}_k,$$

where A is the matrix

$$A = \begin{pmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}.$$

a) Compute three linearly independent eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ of A.

b) Give an explicit formula for $\mathbf{x}_k$ in terms of k, $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$.

c) Compute the limit of $\mathbf{x}_k$ as $k \to \infty$.

10. For any real number k, let $T_k\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map defined by the conditions

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad , \quad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} k^2 - 1 \\ 0 \\ k \end{pmatrix} \quad , \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} k^2 - 2k \\ 0 \\ 0 \end{pmatrix}.$$

Determine the values of k with the following properties:

a) T_k is not invertible. For those values, say what is the rank and the nullity of T_k.

b) T_k preserves distances.

c) T_k has the same rank as T_k^2.

11. Knowing that A is an invertible diagonalizable (4×4)-matrix whose characteristic polynomial is

$$p_A(\lambda) = (\lambda - 5) \cdot (\lambda^2(\lambda - 1) - \lambda + 1),$$

and that P is an invertible (4×4)-matrix, compute the characteristic polynomial $p_B(\lambda)$ of the (4×4)-matrix B given by

$$B = (P(A^T)^2 P^{-1})^{-1}.$$