University of Massachusetts
Department of Mathematics and Statistics
Advanced Exam in Geometry
August 2008

Do 5 out of the following 7 problems. Indicate clearly which questions you want graded. *Passing standard:* 70% with three problems essentially complete. **Justify all your answers.**

1. a) Show that
$$M = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1x_2 + x_3x_4 = 0,\ \ x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\}$$
is a smooth submanifold of $\mathbb{R}^4$.
   b) Show that M is orientable.
   c) Find all critical points of the restriction of the coordinate function x_1 to M.

2. Let M, N be smooth manifolds and $F\colon M \to N$ a submersion. Let Δ be a smooth k-dimensional distribution in N and define:
$$F^*(\Delta)_p \ := \ (F_{*,p})^{-1}(\Delta_{F(p)}) \subset T_p(M).$$
   a) Prove that $F^*(\Delta)$ is a smooth distribution in M. What is the dimension of $F^*(\Delta)$?
   b) Prove that $F^*(\Delta)$ is integrable (involutive) if and only if Δ is integrable (involutive).
   c) Give an example to show that a) may fail if F is not a submersion.

3. A $2n$-dimensional manifold is called symplectic if there exists a closed 2-form ω for which the nth exterior power $\omega \wedge \cdots \wedge \omega$ does not vanish at any point. Prove that
   a) if M is symplectic with symplectic form ω, then $X \mapsto \omega(X, -)$ defines an isomorphism of vector bundles $TM \to T^*M$.
   b) Any orientable, 2-dimensional manifold is symplectic.
   c) For any $n \geq 2$, the sphere S^{2n} is not symplectic but the torus T^{2n} is.

4. Let D be the unit disc in $\mathbb{R}^2$ with the Euclidean metric, and let Δ be the Laplacian with respect to the metric.
   a) Show that for any smooth functions f, g on D,
$$\int_{\partial D} (f\frac{\partial g}{\partial r} - g\frac{\partial f}{\partial r}) = \int_D (f\Delta g - g\Delta f).$$
(here $\partial/\partial r$ is the outward-pointing radial unit vector field, defined on $D \setminus \{(0,0)\}$)
   b) State and prove the generalization of a) to an arbitrary Riemannian manifold.

5. a) Give an example of a non-complete vector field on a manifold.
   b) Show that any left-invariant vector field on a Lie group is complete.

6. Let (M, g) be a Riemannian manifold and let ∇ denote the Riemannian connection. Given a one-form $\alpha \in \mathcal{A}^1(M)$ and a vector field $X \in \mathcal{T}(M)$ define $\nabla_X \alpha$ by
$$\nabla_X \alpha(Y) := X(\alpha(Y)) - \alpha(\nabla_X Y).$$
   a) Prove that $\nabla_X \alpha \in \mathcal{A}^1(M)$.
   b) Prove that
$$\nabla_X(f\alpha) = (Xf)\alpha + f\nabla_X \alpha \ ; \quad f \in C^\infty(M).$$
   c) Suppose $f \in C^\infty(M)$ and $p \in M$ is a critical point of f. Compute $\nabla_X(df)(p)$ in local coordinates $(U, x_1, \ldots, x_n)$ around p.

7. Let $M = (0, \pi/2) \times \mathbb{R} \subset \mathbb{R}^2$ with the Riemannian metric
$$ds^2 = dx^2 + \cos^2(x) dy^2.$$
   a) Compute the Gaussian curvature of (M, ds^2).
   b) Write the geodesic equations for (M, ds^2).