DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS AMHERST
MASTER'S OPTION EXAM — APPLIED MATH
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Do 5 of the following questions. Each question carries the same weight. Passing level is 60% and at least two questions substantially correct.

1. [20 points] Consider the system

$$x' = y\,, \quad y' = -by + x - x^3\,, \tag{1}$$

and consider the quantity $E(x,y) = \frac{1}{2}y^2 - \frac{1}{2}x^2 + \frac{1}{4}x^4$.

(a) Calculate $\frac{d}{dt}E(x,y)$; when (for what value(s) of b) is the system conservative and when dissipative?

(b) Draw the phase portrait of the system in the conservative case and justify carefully your answer; based on the phase plane, sketch all typical "interesting" solutions $x = x(t)$ and $y = y(t)$ as functions of the t variable.

(c) Do the same in the dissipative case.

2. [20 points] Using separation of variables find the series solution (and determine the coefficients) of the pinned square vibrating membrane described by the equation

$$u_{tt} - c^2\big(u_{xx} + u_{yy}\big) = 0\,, \quad 0 < x < 1, \quad 0 < y < 1, \quad t > 0\,,$$

with initial data $u(x,0,t) = 0\,, u(x,1,t) = 0\,, u(0,y,t) = 0\,, u(1,y,t) = 0$ for all $0 < x < 1, \quad 0 < y < 1, \quad t > 0$ and $u_t(x,y,0) = 1$ for all $0 < x < 1, \quad 0 < y < 1$.

3. [20 points] Solve using the method of characteristics the following equation

$$u_t + (x+1)u_x = 0\,, \quad -\infty < x < \infty\,, \quad t > 0\,,$$

with initial datum $u(x,0) = f(x)$.
(b) (10 pts) Can you solve using the method of characteristics,

$$u_t + x^2 u_x = 0\,, \quad -\infty < x < \infty\,, \quad t > 0\,,$$

and initial datum $u(x,0) = f(x)$? Explain the limitations, if any.

4. [20 points] Consider the system

$$x' = x + y + ax(x^2 + y^2)\,, \quad y' = -x + y + ay(x^2 + y^2)\,. \tag{2}$$

(a) Set up the linearized system around the equilibrium point $(0,0)$ and describe its behavior.

(b) How does the original nonlinear system (2) behaves as $t \to \infty$ and for different choices of the constant a, and how do you compare your result to part (a) above.

5. [20 points] Assuming ρ is a postive constant, consider the system

$$x' = x(1 - y)\,, \quad y' = y(\rho - x)\,. \tag{3}$$

(a) Draw the nullclines and base on this calculation sketch the vector field of the system as best you can.

(b) Determine the behavior of the linearized system around the equilibrium points and sketch the local linearized phase plane portraits. Does the value of ρ affect the answers?

(c) Sketch the phase plane portrait of the original nonlinear system (3) and justify your results.

6. [20 points] Solve explicitly the viscous Burgers equation as follows:
(a) Let $u = u(x,t) > 0$ be a solution of the heat equation

$$u_t - ku_{xx} = 0\,, \quad -\infty < x < \infty, \quad t > 0\,.$$

where k is a positive constant. Show that

$$v(x,t) = -\frac{2ku_x(x,t)}{u(x,t)}$$

solves the viscous Burgers equation

$$v_t + vv_x = kv_{xx}\,.$$

(b) Using (a), write an explicit formula for the solution $v = v(x,t)$ of the viscous Burgers equation with initial datum $v(x,0) = \phi(x)$, where ϕ is a smooth function.

7. [20 points] Consider the traffic flow equation

$$\rho_t + [q(\rho)]_x = 0\,, \quad -\infty < x < \infty, \quad t > 0\,,$$

where $\rho = \rho(x,t)$ is the vehicle density at spatial location x and at time t, while $q = q(\rho) = \rho v(\rho)$ is their flux. Furthermore, the average speed $v = v(\rho)$ is given by the constitutive relation

$$v(\rho) = v_m\big(1 - \frac{\rho}{\rho_m}\big)$$

where v_m denotes the speed limit and ρ_m is the maximum density (bumper-to-bumper traffic).

(a) Solve the PDE with initial datum

$$\rho(x,0) = \begin{cases} \rho_m\,, & x \le 0 \\ 0\,, & x > 0 \end{cases}$$

(b) Solve the PDE with initial datum

$$\rho(x,0) = \begin{cases} \frac{1}{8}\rho_m\,, & x \le 0 \\ \rho_m\,, & x > 0 \end{cases}$$

(c) Explain the practical meaning in traffic terms of the two different initial data and accordingly interpret the corresponding solutions in parts (a) and (b).