DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS AMHERST
APPLIED MATHEMATICS EXAM
Spring 2023

Do all six problems. All problems carry equal weight.
Passing level: 65% with at least three substantially correct.

1 The KPP or Fisher equation

$$u_t = u_{xx} + u(1-u)$$

is a PDE that describes the diffusion of a spatially distributed population with logistic growth. Traveling wave solutions $u = u(x - ct)$ satisfy the ODE

$$u'' + cu' + u(1-u) = 0.$$

(a) Convert the above ODE into a first-order ODE system;

(b) Find the equilibrium points of the ODE system;

(c) Linearize the ODE system at these equilibrium points;

(d) Solve the eigenvalues of the two linearized systems. And classify these equilibrium points, according to different c values.

2 Consider equation

$$x' = \mu x + x^3 - x^5\,,$$

where $\mu \in \mathbb{R}$ is a parameter.

(a) Identify the bifurcation points as functions of μ and classify their stability. How would the system behave if μ is increased quasi-statically from $-\infty$ to $+\infty$? When is the equilibrium $x = 0$ linearly stable but unstable to sufficiently large perturbations?

(b) Sketch the bifurcation diagram. Find and classify the bifurcation point.

3 Consider the nonlinear system

$$dx/dt = -x + y + 3y^2, \qquad dy/dt = y.$$

(a) Sketch the phase plane, and show that its flow map $\varphi_t : \mathbb{R}^2 \to \mathbb{R}^2$ is given by

$$\varphi_t(x, y) = \left(xe^{-t} + y\sinh t + y^2(e^{2t} - e^{-t}), ye^t\right).$$

Hint. You can first solve for y, and then plug it into the equation of dx/dt.

What are the stable and unstable manifolds of $(0, 0)$?

(b) Linearize the system at the equilibrium $(x, y) = (0, 0)$. Classify the equilibrium, sketch the phase plane of the linearized system, and show that its flow map $e^{tA} : \mathbb{R}^2 \to \mathbb{R}^2$ is given by

$$e^{tA} = \begin{pmatrix} e^{-t} & \sinh t \\ 0 & e^t \end{pmatrix}.$$

What are the stable and unstable subspaces?

4 (a) An $n \times n$ matrix N is said to be nilpotent if $N^k = 0$ for some $k \in \mathbb{N}$. Compute e^{tA} if $A = \lambda I + N$ where N is nilpotent. Justify all your steps.

(b) Compute e^{tA} if A is the 3×3 Jordan block

$$A = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}.$$

(c) Consider the 3×3 linear system $x_t = Ax$ where A is the matrix in (b) and $\lambda \in \mathbb{R}$. Describe the stable, unstable, and center subspaces and the stability of $x = 0$ in the cases (i) $\lambda < 0$; (ii) $\lambda = 0$; (iii) $\lambda > 0$.

5 A model for a population $x(t) \geq 0$ with logistic growth and a constant rate of harvesting is

$$x_t = \mu x \left(1 - \frac{x}{K}\right) - H$$

where the parameters μ, K, H are positive constants.

(a) Show that a nondimensionalized form of the equation is

$$x_t = x(1 - x) - h, \tag{1}$$

and express the dimensionless parameter h as a ratio of two times. (b) Sketch a graph of the equilibria as functions of h, and sketch the phase line of (1) for various values of $h > 0$. Determine the stability of the equilibria, both from the phase line and from their linearized stability. For what values of the initial (nondimensionalized) population $x_0 > 0$ and harvesting rate $h > 0$ does the population become extinct?

6 In modeling the digestion process in insects, it is believed that digestion yield rate Y, in mass per time, is related to the concentration C of the limiting nutrient, the residence time T in the gut, the gut volume V, and the rate of the nutrient breakdown r, given in mass per time per volume. Show that the yield is positively related to the gut volume, if T, r, C are fixed.