Analysis Qualifying Examination

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This exam consists of eight equally weighted problems (ten points each): a passing grade is 65% (52/80), including at least five "essentially correct" problems ($\approx 7.5/10$).

Clearly show your work, explicitly stating or naming results that you use; justify the use of named theorems by verifying the necessary conditions.

Please work legibly and clearly. Put your name at the top of this page.

Unless otherwise stated, the measure in every problem is the Lebesgue measure. Unless otherwise specified, the underlying space is $\mathbb{R}^d$.

1. Let $f : \mathbb{R} \to \mathbb{R}$. Prove that f satisfies the Lipschitz condition

$$|f(x) - f(y)| \leq M|x - y|$$

for some M and all $x, y \in \mathbb{R}$, if and only if f satisfies the following two properties

   (a) f is absolutely continuous.

   (b) $|f'(x)| \leq M$ for a.e. x.

2. Let $T : H \to H$ be a compact operator on a Hilbert space H, and let λ be a nonzero number.

   (a) Prove that that there is $c > 0$ such that $\|(T - \lambda I)f\| \geq c\|f\|$ for all f that are orthogonal to $\mathrm{Ker}(T - \lambda I)$. Hint: Argue by contradiction.

   (b) Prove that the range of $T - \lambda I$ defined by

   $$\{g \in H \ : \ g = (T - \lambda I)f, \text{ for some } f \in H\}$$

   is closed. Hint: Suppose $g_j \to g$ where $g_j = (T - \lambda I)f_j$. Argue that f_j may be taken to be orthogonal to $\mathrm{Ker}(T - \lambda I)$ and that under this assumption $\{f_j\}$ is a bounded sequence. You may use part (a).

   (c) Prove that the range of $T - \lambda I$ is all of H if and only if the kernel of $\overline{\lambda} I - T^*$ is trivial. Hint: You may use part (c).

3. Suppose ν, ν_1, ν_2 are signed measures on $(M, \mathcal{F})$ and μ a positive measure on $(M, \mathcal{F})$. Prove:

   (a) If $\nu_1 \perp \mu$ and $\nu_2 \perp \mu$ then $\nu_1 + \nu_2 \perp \mu$.

   (b) If $\nu_1 \ll \mu$ and $\nu_2 \ll \mu$ then $\nu_1 + \nu_2 \ll \mu$.

   (c) $\nu_1 \perp \nu_2$ implies $|\nu_1| \perp |\nu_2|$.

   (d) $\nu \ll |\nu|$.

   (e) If $\nu \perp \mu$ and $\nu \ll \mu$ then $\nu = 0$.

4. You may use the conclusion of part (a) in proving part (b).

   (a) Let φ be a compactly supported smooth function supported in the ball of radius one, and let $\varphi_k(\xi) = \varphi(2^k \xi)$. With $\mathcal{F}$ and $\mathcal{F}^{-1}$ denoting the Fourier transform and the inverse Fourier transform respectively, for any Schwartz function f let $T(f) := \mathcal{F}^{-1}(\varphi_k \mathcal{F}(f))$. Prove that there is a constant C, independent of f and k, such that for $1 \leq p \leq q < \infty$

$$\|T(f)\|_{L^q} \leq C 2^{k(\frac{d}{p} - \frac{d}{q})} \|f\|_{L^p}.$$

   Hint: You may use Young's inequality $\|f * g\|_{L^r} \leq \|f\|_{L^p} \|g\|_{L^q}$, when $1 \leq p, r, q \leq \infty$ satisfy $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$.

   (b) With the same notation as in the previous part prove that there is a constant C, independent of f and k, such that for $1 \leq p \leq q < \infty$

$$\|T(f)\|_{L^q} \leq C 2^{k(\frac{d}{p} - \frac{d}{q})} \|T(f)\|_{L^p}.$$

   Hint: Think of another function like φ but with slightly different support properties (applied to T on the Fourier side).

5. Prove that if f is integrable on $\mathbb{R}^d$ and f is not identically zero, then (here f^* denotes the maximal function)

$$f^*(x) \geq \frac{c}{|x|^d}, \text{ for some } c > 0 \text{ and all } |x| \geq 1.$$

Conclude that f^* is not integrable on $\mathbb{R}^d$. Then prove that the weak type estimate

$$m(\{x \ : \ f^*(x) > \alpha\}) \leq c/\alpha$$

for all $\alpha > 0$ whenever $\int |f| = 1$ is optimal in the following sense: if f is supported in the unit ball with $\int |f| = 1$then

$$m(\{x \ : \ f^*(x) > \alpha\}) \geq c'/\alpha$$

for some $c' > 0$ and all sufficiently small α.

6. Let E be a subset of $\mathbb{R}^d$. Egorov's theorem states that if $m(E) < \infty$ and f, $\{f_k\}_{k\geq 1}$ are measurable functions on E such that $f_k \to f$ a.e. in E then for every $\epsilon > 0$ there exists a closed set A_ϵ such that $m(E \setminus A_\epsilon) \leq \epsilon$ and $f_k \to f$ uniformly on A_ϵ. The type of convergence involved in the conclusion of Egorov's theorem is sometimes called almost uniform convergence. For the questions below we assume that we are under the hypothesis of Egorov.

   (a) Give an example to show that under the hypothesis of Egorov one cannot conclude that $f_k \to f$ uniformly on $E \setminus Z$ where $m(Z) = 0$.

   (b) Prove that if $f_k \to f$ almost uniformly in E then $f_k \to f$ a.e. in E.

   (c) Prove that if $f_k \to f$ almost uniformly in E then for every $\delta > 0$ we have that $m(\{x \ : \ |f_k(x) - f(x)| \geq \delta\}) \to 0$ as $k \to \infty$. Hint: Fix a given $\delta > 0$ and use the hypothesis to prove that for every $\epsilon > 0$ there exists a K such that for all $k \geq K$, $m(\{x \ : \ |f_k(x) - f(x)| \geq \delta\}) \leq \epsilon$.

7. Suppose that $\{f_n\}_{n\geq 1}$ is a sequence of non-negative measurable functions that converges in measure to f. Show that

$$\int f dx \leq \liminf_n \int f_n(x) dx.$$

Hint: In the course of the proof you may use the fact that if a sequence of measurable functions $\{g_n\}$ converges in measure to another measurable function g, then there is a subsequence $\{g_{n_j}\}$ that converges almost everywhere to g.

8. We say that the family of integrable functions $K_\delta : \mathbb{R}^d \to \mathbb{R}$, indexed by δ, is an approximation to the identity if

- $\int_{\mathbb{R}^d} K_\delta(x)dx = 1$.
- $|K_\delta(x)| \leq C\delta^{-d}$ for some $C > 0$ and all $\delta > 0$.
- $|K_\delta(x)| \leq C\delta/|x|^{d+1}$ for some $C > 0$ and all $\delta > 0$ and all $x \in \mathbb{R}^3$.

(a) Prove that if $\{K_\delta\}_\delta > 0$ is an approximation to the identity and f is integrable of $\mathbb{R}^3$, then

$$(f * K_\delta)(x) \to f(x) \quad \text{as } \delta \to 0$$

for every x in the Lebesgue set of f. Hint: First show that for all x in the Lebesgue set of f, $A(r) = \frac{1}{r^d} \int_{|y| \leq r} |f(x-y) - f(x)|dy$, defined for $r > 0$, is continuous, bounded, and satisfies $A(r) \to 0$ as $r \to 0$.

(b) Verify that $\{P_y(x)\}_{y>0}$ defined by $P_y(x) = \frac{1}{\pi} \frac{y}{x^2+y^2}$, $x \in \mathbb{R}$, is an approximation to the identity.