

DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS, AMHERST
ADVANCED EXAM — ALGEBRA. SPRING 2023

Passing Standard: The passing standard is 70% with essentially correct solutions to five problems. Show all your work and justify your answers carefully. All rings contain the identity and all ring homomorphisms preserve the identity.

PART I. GROUP THEORY

1. (a) Let G be a group, and let H be a subgroup of finite index. Show that G has a normal subgroup N with finite index such that $N \subset H$.

(b) Let G be a group, and let H_1, H_2 be subgroups of finite index. Show that $H_1 \cap H_2$ also has finite index in G .

Please work out each problem on a separate sheet of paper.

2. Show that there are exactly two isomorphism classes of non-Abelian groups of order 8. Describe these two groups in terms of generators and relations. *Show your work!*

PART II. COMMUTATIVE ALGEBRA

3. Let k be a field of characteristic zero, and let M be an $n \times n$ matrix over k . Show that M is nilpotent (i.e. $M^\ell = 0$ for some integer $\ell > 0$) if and only if $\text{trace}(M^i) = 0$ for every $1 \leq i \leq n$.

Please work out each problem on a separate sheet of paper.

4. Denote by \mathbb{F}_{p^n} the finite field of size p^n ; it is naturally a \mathbb{F}_p -module.

(a) For any positive integers m, n , determine

$$\mathbb{F}_{p^n} \otimes_{\mathbb{F}_p} \mathbb{F}_{p^m}$$

as a \mathbb{F}_p -module. Show your work!

(b) With respect to the multiplicative law

$$(a_1 \otimes b_1) \cdot (a_2 \otimes b_2) \mapsto (a_1 a_2) \otimes (b_1 b_2),$$

under what condition on m, n would

$$\mathbb{F}_{p^n} \otimes_{\mathbb{F}_p} \mathbb{F}_{p^m}$$

be a field? Justify!

Please work out each problem on a separate sheet of paper.

5. Let $A \subset B$ be an integral ring extension. Suppose $B \setminus A$ is multiplicatively closed in B . Show that A is integrally closed in B .

PART III. FIELD THEORY AND GALOIS THEORY

6. Let K/\mathbf{Q} be a finite field extension. Show that K contains only finitely many roots of unity.

Please work out each problem on a separate sheet of paper.

7. Let k be a field, and let $f \in k[x]$ be a separable polynomial of degree n . Let L/k be a splitting field for f , and denote by c_1, \dots, c_n the roots of f in L . It is a standard fact that L/k is a Galois extension, and that $\text{Gal}(L/k)$ permutes these roots.

(a) Show that the $\text{Gal}(L/k)$ -action on the roots c_1, \dots, c_n is transitive if and only if f is k -irreducible.

(b) The $\text{Gal}(L/k)$ -action on these roots allows us to view $\text{Gal}(L/k)$ as a subgroup of the symmetric group S_n . Show that if n is prime, then $\text{Gal}(L/k)$ contains an n -cycle.

Passing Standard: The passing standard is 70% with essentially correct solutions to five problems. Show all your work and justify your answers carefully. All rings contain the identity and all ring homomorphisms preserve the identity.
