

Basic Exam: Advanced Calculus & Linear Algebra

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Name: _____

Instructions: Do all eight problems and show your work. Justify all your answers. If you run out of space of the same page, continue on the back of the page and clearly indicate that you have done so. Write your name on each page. The passing standards are:

- Master's level: 60% with three questions essentially complete, including one question from each part;
- Ph.D. level: 75% with two questions from each part essentially complete.

The first part of the exam covers Advanced Calculus. The second part covers Linear Algebra.

Name:

Calculus

1. What fraction of the area of the surface of the sphere S given by $x^2 + y^2 + z^2 = 4$ lies above the plane $z = 1$?

Name:

Calculus

2. Let $C(H, R)$ be the cone of height $H > 0$ and radius $R > 0$.

- (a) Show that the volume $V(H, R)$ of the cone $C(H, R)$ is $\frac{\pi HR^2}{3}$.
- (b) Use the method of Lagrange multipliers to maximize the volume $V(H, R)$ when subject to the constraint $H + R = 9$.
- (c) Show that for every $\varepsilon > 0$ there exists $H_0 > 0$ and $R_0 > 0$ such that $H_0 + R_0 = 9$ and $V(H_0, R_0) < \varepsilon$.

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Calculus

3. Prove that

$$\int_0^1 \int_0^1 \frac{1}{1-xy} dx dy = \sum_{k=1}^{\infty} \frac{1}{k^2}.$$

Hint: write the integrand as a geometric power series.

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Calculus

4. Let Σ be a smooth oriented surface in \mathbb{R}^3 with boundary $\partial\Sigma$, and let φ and ψ be smooth functions on \mathbb{R}^3 . Denote by $\nabla\varphi$ and $\nabla\psi$ their gradient vector fields, and by $\nabla\varphi \times \nabla\psi$ their vector product.

(a) Show that

$$\iint_{\Sigma} (\nabla\varphi \times \nabla\psi) \cdot d\mathbf{S} = \int_{\partial\Sigma} (\varphi\nabla\psi) \cdot d\mathbf{r}.$$

(b) Show that

$$\int_{\partial\Sigma} (\varphi\nabla\psi + \psi\nabla\varphi) \cdot d\mathbf{r} = 0.$$

You can use without proving it that, if \vec{G} is a smooth vector field on \mathbb{R}^3 , we have the “Leibnitz rule”

$$\nabla \times (\varphi\vec{G}) = \nabla\varphi \times \vec{G} + \varphi(\nabla \times \vec{G}).$$

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Linear Algebra

5. Consider the linear system

$$\begin{aligned}x_1 + x_2 + x_3 + x_4 &= 0 \\x_1 - x_2 - x_3 + 5x_4 &= 0 \\2x_2 + 2x_3 - 4x_4 &= 0\end{aligned}$$

- (a) Let V be the solution set to the linear system. Find a basis for V .
- (b) Find a basis for V^\perp , the orthogonal complement to V .

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Linear Algebra

6. Let $UT_2(\mathbb{R})$ be the vector space of all upper triangular (2×2) -matrices with entries in \mathbb{R} .

The *standard basis* for $UT_2(\mathbb{R})$ is the set

$$\mathcal{B}_1 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

You may use without proof that this is a basis for $UT_2(\mathbb{R})$.

(a) Prove that the set

$$\mathcal{B}_2 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix} \right\}$$

is also a basis for $UT_2(\mathbb{R})$.

(b) Find the change-of-basis matrix from \mathcal{B}_1 to \mathcal{B}_2 and use it to write the matrix $\begin{pmatrix} 6 & 4 \\ 0 & 5 \end{pmatrix}$ as a linear combination of the elements in \mathcal{B}_2 .

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Linear Algebra

7. For any real number k , consider the (4×2) -matrix

$$M_k = \begin{pmatrix} 1 & k-5 \\ 0 & 10-k \\ 1 & 5-k \\ -k-3 & 0 \end{pmatrix}.$$

- (a) For every value of k find a (2×4) -matrix B_k such that $B_k M_k$ is the identity matrix I_2 . *Hint: think about how the columns of M_k are related to each other.*
- (b) Show that for every value of k there exists no (2×4) -matrix A_k such that $M_k A_k$ is the identity I_4 .

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Linear Algebra

8. Let $n > 0$. Let \mathbb{P}_n denote the vector space of polynomials of degree n or smaller. Consider the linear transformation $T_n: \mathbb{P}_n \rightarrow \mathbb{P}_n$ that sends a polynomial to its first derivative. Let A_n be the $(n+1) \times (n+1)$ -matrix that represents the linear transformation T_n with respect to the canonical basis $\mathcal{B}_n = \{1, t, t^2, \dots, t^n\}$ of \mathbb{P}_n .
- (a) Determine whether A_n is diagonalizable.
 - (b) Determine the Jordan normal form and a basis of generalized eigenvectors for A_n .

Justify all your answers. *Note: You are being asked to solve this exercise for arbitrary $n > 0$. But, if you're stuck, for partial credit you can solve the cases $n = 2$ and $n = 3$ at least, and guess the answer for general n .*