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Advanced Probability Qualifying Examination
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Friday, September 1, 2023

Instructions

1. This exam consists of six (6) problems (each of equal weight 20). You need to solve 5 out of 6 problems and your grade will be evaluated using the five problems you choose (or the best five out of six problems if you decide to solve all the problems).

2. In order to pass this exam, it is enough that you solve essentially correctly at least three (3) problems and that you have an overall score of at least 65%.

3. State explicitly all results that you use in your proofs and verify that these results apply.

4. Please write your work and answers <u>clearly</u> in the blank space under each question.

5. The last page is empty and can be used if you need more space.

1. Consider the function $F : \mathbb{R} \to \mathbb{R}$ given by

$$F(y) = \exp\left(-e^{-y}\right).$$

   (a) Show that F is a (cumulative) distribution function for a random variable Y, called the Gumbel random variable.

   (b) Let $\{X_n\}_{n=1}^{\infty}$ be i.i.d. exponential random variables with parameter $\lambda = 1$, i.e.

$$\mathbb{P}(X_n \leq x) = \begin{cases} 1 - e^{-x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

   Let $M_N := \max\{X_1, \ldots, X_N\}$. Show that the random variable $M_N - \log(N)$ converges in distribution to a Gumbel random variable.

2. Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space.

   (a) Suppose that the random variable $X : \Omega \to \mathbb{R}$ is non-negative, i.e. $X \geq 0$ almost surely. Prove that
   $$\mathbb{E}[X] = \int_0^\infty \mathbb{P}[X > t]\, dt.$$

   (b) Suppose that the random variable $X : \Omega \to \mathbb{N}$ take values in the non-negative integers. Prove that
   $$\mathbb{E}[X] = \sum_{k=0}^\infty \mathbb{P}[X > k].$$

3. Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space.

   (a) Prove the Borel–Cantelli lemma. That is, show that if $\{A_n; n \in \mathbb{N}\}$ is a sequence of sets in $\mathscr{F}$ with $\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty$, then $\mathbb{P}[\limsup_n A_n] = 0$.

   (b) Let $X : \Omega \to \mathbb{R}$ and $\{X_n : \Omega \to \mathbb{R}; n \in \mathbb{N}\}$ be random variables. Suppose that for any $\varepsilon > 0$, we have

$$\sum_{n=1}^{\infty} \mathbb{P}(|X_n - X| \geq \varepsilon) < \infty.$$

   Show that $X_n \xrightarrow{\text{as}} X$.

4. For $1 \le p < \infty$ let

$$L^p = L^p((\Omega, \mathscr{F}, \mathbb{P})) = \{X : \Omega \to \mathbb{R} \text{ Borel measurable}\,;\ \mathbb{E}[|X|^p] < \infty\}$$

and denote $\|X\|_p = E[|X|^p]^{\frac{1}{p}}$.

(a) For $1 \le p < q < \infty$, show that if $X \in L^q$, then $X \in L^p$.

(b) For $1 \le p < q < \infty$, give an example of a random variable X so that $X \in L^p$, but $X \notin L^q$.

(c) Show that if $\{p_n\}$ is an increasing sequence with $\lim_{n\to\infty} p_n = q$, then $\lim_{n\to\infty} \|X\|_{p_n} = \|X\|_q$.

5. Suppose X_n is a Markov chain with state space $S = \{1, 2, \ldots, N\}$ and transition matrix $P(i, j)$. Let $c : S \to \mathbb{R}$ be given. For $\alpha \in (0, 1)$, define $\phi : S \to \mathbb{R}$ by

$$\phi(i) = \mathbb{E}\left[\sum_{n=0}^{\infty} \alpha^n c(X_n) | X_0 = i\right].$$

(a) Show that the expectation defining ϕ exists and that ϕ is finite.

(b) Show that ϕ is the solution of

$$\phi = \alpha P \phi + c.$$

(c) Show that the solution in part (b) is unique.

6. Suppose N_t is a Poisson process with rate λ.

   (a) Show that $N_t - \lambda t$ is a martingale with respect to the filtration $\{N_s\}_{s \geq 0}$.

   (b) Compute the covariance of the process, i.e. the function

   $$c(t, s) = \mathrm{Cov}(N_t, N_s).$$

   (c) Show that conditioned on the event $\{N_t = 1\}$, the time of the first event of the process N_t is uniformly distributed on $[0, 1]$.