UNIVERSITY OF MASSACHUSETTS
Department of Mathematics and Statistics
Basic Exam - Probability
Monday, August 28, 2023

Work all problems. 60 points are needed to pass at the Masters Level and 75 to pass at the Ph.D. level.

1. Suppose X has a *truncated* Poisson(λ) distribution where $X = 0$ cannot be observed, but the relative probabilities of other values are proportional to the Poisson mass function, $f(x) = \lambda^x \exp(-\lambda)/x!, x = 0, 1, 2, \ldots$
   (a) Find the probability mass function of X.
   (b) Derive the mean of X.
   (c) Derive the variance of X.

2. Suppose $U_1, \ldots, U_n \overset{\text{iid}}{\sim}$ uniform$(0, 1)$. This means the probability density function (PDF) of U_1 is
$$f_{U_1}(u) = \begin{cases} 1 & u \in [0,1] \\ 0 & u \notin [0,1]. \end{cases}$$
   Let $S_n = \sum_{k=1}^{n} U_k$.
   (a) Let F_{S_n} denote the cumulative distribution function (CDF) of S_n. Use induction to show that for $s \in [0, 1]$,
$$F_{S_n}(s) = \frac{s^n}{n!}.$$
   (b) Find M_{S_n}, the moment-generating function of S_n.
   (c) Use the MGF to find $E(S_n)$.
   (d) Use a method of your choosing to find $\text{Var}(S_n)$.

3. A real-valued random variable X is said to follow the log-normal distribution with parameters $\mu \in \mathbb{R}$ and $\sigma \in (0, \infty)$ if it has density function $f(x; \mu, \sigma)$ given by
$$f(x; \mu, \sigma) = \frac{1}{x\sigma\sqrt{2\pi}} \exp\left\{-\frac{(\ln(x) - \mu)^2}{2\sigma^2}\right\}$$
   for $x > 0$, and 0 otherwise.

   Suppose that $X_1, X_2, \ldots, X_n$ is a sequence of IID log-normal(μ, σ) random variables.
   (a) Show that $E[X_i] = \exp\{\mu + \sigma^2/2\}$ and $\text{Var}(X_i) = \exp\{2\mu + 2\sigma^2\} - \exp\{2\mu + \sigma^2\}$ (hint: use the change of variables $y = (\ln(x) - \mu)/\sigma$).
   (b) Define the sample mean $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$. Show that $\bar{X}_n$ converges in probability to a constant c, and find c (which may depend on μ and/or σ).

(c) Show that $\sqrt{n}(\bar{X}_n - c)$ converges in distribution, and find the limit distribution.

(d) Suppose that σ is known. Show that $\ln(\bar{X}_n) - \sigma^2/2$ converges in probability to a constant c', and find c' (which may depend on μ and/or σ).

(e) Show that $\sqrt{n}(\ln(\bar{X}_n) - \sigma^2/2 - c')$ converges in distribution, and find the limit distribution.

4. Let X and Y be random variables with pdf: $f(x, y) = 1$ when $0 \leq x \leq 1, x \leq y \leq x+1$, and 0 otherwise.

   (a) Show that $f(x, y)$ is a density.

   (b) Are X and Y independent? Why or why not?

   (c) Find the marginal density of Y.

   (d) Find $E(Y|X = x)$.

   (e) Find $Pr(X + Y < 0.5)$.