Analysis Qualifying Examination

Department of Mathematics and Statistics
University of Massachusetts

August 2023

This exam consists of eight equally weighted problems (ten points each): a passing grade is 65% (52/80), including at least five “essentially correct” problems ($\gtrsim 7.5/10$).

Clearly show your work, explicitly stating or naming results that you use; justify the use of named theorems by verifying necessary conditions.

Please work legibly and clearly label each page/file of your exam with your name. You may use the back of the page for additional work.

1. Calculate the following limits and justify your calculations:

   (a) $\lim_{n\to\infty} \int_0^\infty (1 + (x/n))^{-n} \sin(x/n) dx$.

   (b) $\lim_{n\to\infty} \int_0^1 (1 + nx^2)(1 + x^2)^{-n} dx$.

   (c) $\lim_{n\to\infty} \int_0^\infty n \sin(x/n)[x(1 + x^2)]^{-1} dx$.

2. Let $(X, \mathcal{M}, \mu)$ be a finite measure space with $\mu(X) > 0$ and let φ be a nonnegative, Borel-measurable function mapping X into $[0, \infty)$. Assume that $\varphi \not\equiv 0$ a.e. For $A \in \mathcal{M}$, define

$$\lambda(A) = \int_A \varphi \, d\mu \, .$$

   (a) Prove that there exists $A \in \mathcal{M}$ such that $\lambda(A) > 0$. This proves that λ is nontrivial.

   (b) Prove that λ is a measure on $\mathcal{M}$ and that $\lambda \ll \mu$.

   (c) Let h be any nonnegative, Borel-measurable function mapping X into $[0, \infty)$. Prove that

$$\int_X h \, d\lambda = \int_X \varphi \, h \, d\mu \, .$$

3. If $a, b > 0$, let

$$f(x) = \begin{cases} x^a \sin(x^{-b}) & \text{for } 0 < x \leq 1 \\ 0 & \text{if } x = 0 \end{cases}.$$

Prove that f is of bounded variation in $[0, 1]$ if and only if $a > b$.

4. Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be two measure spaces and $K, K_1, K_2 : X \times Y \to \mathbb{R}$ such that $|K| \leq K_1 K_2$ and

$$\|K_1(x, \cdot)\|_{L^q(Y)} \leq C_1$$

μ almost everywhere and

$$\|K_2(\cdot, y)\|_{L^p(X)} \leq C_2$$

ν almost everywhere, with $\frac{1}{p} + \frac{1}{q} = 1$, $p, q, \in (1, \infty)$. Prove that the linear operator

$$(Tf)(x) = \int_Y K(x, y) f(y) d\nu(y)$$

is bounded from $L^p(Y)$ to $L^p(X)$, with $\|T\| \leq C_1 C_2$.

5. Let H be an infinite dimensional Hilbert space with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$.

   (a) Prove that the unit sphere in H (that is $\{x \in H \mid \|x\| = 1\}$) is not compact.

   (b) Show that if $\{x_n\}$ is a sequence of unit vectors in H then there is a subsequence $\{x_{n_j}\}$ and an element $x \in H$ such that for all $y \in H$

   $$\lim_{j\to\infty} \langle x_n, y\rangle = \langle x, y\rangle.$$

   Hint: Let y run through a basis for H and use a diagonalization argument. One can then defined x by giving its series expansion with respect to the chosen basis.

6. In this problem $\hat{f}$ denotes the Fourier transform of $f : \mathbb{R}^d \to \mathbb{R}$.

(a) Prove that if $f \in L^1$ then $\hat{f}$ is continuous and $\lim_{|\xi| \to \infty} \hat{f}(\xi) = 0$.

(b) Let $d = 1$ (that is, $f : \mathbb{R} \to \mathbb{R}$). Prove that there is a constant $C > 0$ such that for any Schwartz function f

$$\Big(\int_{\mathbb{R}} x^2 |f(x)|^2 dx \Big) \Big(\int_{\mathbb{R}} \xi^2 |\hat{f}(\xi)|^2 d\xi \Big) \geq C \|f\|_{L^2}^2.$$

Hint: Note that $|f(x)|^2 = |f(x)|^2 \frac{d}{dx} x$, and use integration by parts.

7. Let H be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$, and recall that a bounded linear operator $T : H \to H$ is called compact if for every bounded sequence $\{x_n\}$ in H, $\{Tx_n\}$ has a convergent subseqeunce.

   (a) Give an example of a compact operator and an example of a non-compact operator, both defined on an infinite dimensional Hilbert space.

   (b) Prove that if T is compact then so are T^* and T^*T.

   (c) Prove that if T is compact then there exists orthonormal sets (neither of which is necessarily a basis) $\{\phi_n\}_{n=1}^N$ and $\{\psi_n\}_{n=1}^N$ (N may be a positive integer or ∞) and positive real numbers $\{\lambda_n\}_{n=1}^N$ with $\lambda_n \to 0$ such that

   $$T = \sum_{n=1}^{N} \lambda_n \langle \psi_n, \cdot \rangle \phi_n.$$

   Hint: Use the previous part (even if you couldn't prove it) and the spectral theorem for compact self-adjoint operators.

8. Let $(X, \mathcal{A}, \mu)$ be a measure space. Let f and f_n, $n \geq 1$ be measurable functions on X. Recall that f_n is said to converge to f in measure if for every $\epsilon > 0$,

$$\mu(\{x \mid |f_n(x) - f(x)| \geq \epsilon\}) \to 0 \qquad \text{as } n \to \infty.$$

(a) Suppose that $\mu(X) < \infty$. Prove that $f_n \to f$ a.e implies $f_n \to f$ in measure.

(b) Prove that the converse of (b) is false even for $\mu(X) < \infty$. (**Hint.** Let $X = [0, 1]$ and consider the (double) sequence $f_{m,k}(x) = 1_{E_{m,k}}(x)$ $(m, k \in \mathbb{N})$, where
$E_{m,k} := [\frac{m-1}{k}, \frac{m}{k}]$.)