

Sample algebraic topology qualifying exam questions

The new topology qualifying exam began in 2018. In addition to questions on point-set topology, students taking the new topology exam are expected to be familiar with the following topics:

- Definition and elementary properties of homotopy; homotopy equivalences; deformation retracts.
- The definition of π_1 ; functoriality under mappings and invariance under homotopy. The relation between π_1 at different base points. The fundamental group of a cartesian product.
- The path lifting/homotopy lifting lemmas, their proofs, and their use in proving that $\pi_1(S^1) \cong \mathbb{Z}$.
- The Seifert-van Kampen theorem, and its use in computing π_1 of various spaces, such as compact surfaces.
- Covering spaces; path and homotopy lifting theorems; classification of connected covers via subgroups of the fundamental group.
- Cell complexes, Δ -complexes and simplicial complexes, the classification of compact surfaces.
- Singular, simplicial and cellular homology; Hurewicz theorem relating H_1 and π_1 ; degree of maps between spheres (and connected orientable manifolds), induced homomorphisms, homotopy invariance; reduced homology; relative homology; long exact sequences of a pair, a triple, and the Mayer-Vietoris sequence; excision; Homology with coefficients, the universal coefficients theorem; Euler characteristic.
- Simplicial, singular and cellular cohomology; the cup product; Künneth theorems; orientations, the cap product and Poincaré duality.

Here are some sample questions covering these topics, of a level that could appear on the new qualifying exam.

- (1) Let X be the quotient of the closed unit disk in \mathbb{C} by the equivalence relation given by $z \sim w$ if $z = w$ or if $|z| = 1$ and $z = e^{2\pi k/3}w$ for some $k \in \mathbb{Z}$. Compute the fundamental group of X .
- (2) Let $f, g : X \rightarrow Y$ continuous maps between topological spaces. Prove or disprove that f and g must be homotopic if
 - (a) Y is contractible
 - (b) X is contractible
 - (c) Y is simply connected and $X = S^1$.

- (3) Give an example of a pair of spaces $X \subset Y$ where X is a retract of Y but not a deformation retract. Give an example where X is deformation retract of Y but not homeomorphic to Y .
- (4) Show that a Möbius strip does not retract onto its boundary circle.
- (5) Given a topological space X , its suspension SX is the space obtained from $X \times [0, 1]$ by identifying the end $X \times \{0\}$ to a point, and the end $X \times \{1\}$ to another point.
- Prove that if X is path connected, then SX is simply connected.
 - Prove that if X is contractible, then SX is also contractible.
 - Use a Mayer-Vietoris sequence to compute the homology groups of SX in terms of the homology groups of X .
- (6) Let X is a path-connected subset of \mathbb{R}^2 which contains the unit circle but does not contain the origin. Prove that $\pi_1(X)$ contains a subgroup isomorphic to \mathbb{Z} .
- (7)
 - Show that $S^1 \times S^1$ and $S^1 \vee S^1 \vee S^2$ have isomorphic homology groups in all degrees, but their universal covers do not.
 - Show that any map $S^2 \rightarrow S^1 \times S^1$ is nullhomotopic.
 - Find a map $S^2 \rightarrow S^1 \vee S^1 \vee S^2$ which is not nullhomotopic.
- (8) Give examples of normal and non-normal 3-sheeted covering spaces of $S^1 \vee S^1$.
- (9) Let $X = S^2/\{p, q\}$ be the two-sphere with two points identified.
- Compute the local homology groups $H_*(X, X \setminus \{x\})$
 - when $x = [p]$ is the image of p or q in X , and
 - when x is any other point.
 - Deduce that a homeomorphism of X with itself must take $[p]$ to itself.
- (10) Let T_1, T_2 be two copies of the solid torus $S^1 \times D^2$. Given a homeomorphism $\phi: \partial T_1 \rightarrow \partial T_2$, let X_ϕ be the quotient space obtained by from the disjoint union of T_1 and T_2 by identifying x with $\phi(x)$ for every $x \in \partial T_1$. Find two non-homeomorphic spaces that can be obtained this way, and prove that your spaces are not homeomorphic.

- (11) Given a map $\phi: X \rightarrow X$ of a space X to itself, the *mapping torus* T_ϕ is the quotient of $X \times [0, 1]$ by the equivalence generated by $(x, 1) \sim (\phi(x), 0)$ for all $x \in X$. Compute the homology groups $H_*(T_\phi)$ when (a) $X = S^n$ and ϕ has degree d , and (b) when $X = S^n \vee S^n$ and ϕ interchanges the two spheres (so that ϕ^2 is the identity).
- (12) Let X be the space obtained by gluing two copies of S^2 along their equatorial S^1 (using the identity map). Calculate the homology groups (with integer coefficients) of X . Call one of the spheres A , and the other B . Write down the long exact sequence of homology groups (with integer coefficients) for the pair (X, A) , and calculate every group in this sequence.
- (13) Give an example of a space X and a map $\phi: S^1 \rightarrow X$ such that the induced homomorphism $\phi_*: H_1(S^1) \rightarrow H_1(X)$ is trivial, but the induced homomorphism on π_1 is not.
- (14) Consider maps $f: S^1 \vee S^1 \rightarrow T^2$ and $g: T^2 \times S^1 \vee S^1$. Is it possible for $f \circ g$ to be homotopic to the identity? Is it possible for $g \circ f$ to be homotopic to the identity? Justify your answers.
- (15) Let M_g be a closed orientable surface of genus $g \geq 1$. Show that for each nonzero $\alpha \in H^1(M; \mathbb{Z})$ there exists $\beta \in H^1(M; \mathbb{Z})$ with $\alpha \cup \beta \neq 0$. Deduce that M is not homotopy equivalent to a wedge sum $X \vee Y$ of CW complexes both of which have non-trivial reduced homology. Do the same for closed nonorientable surfaces using cohomology with \mathbb{Z}_2 coefficients.
- (16) Let M be a closed, connected, orientable n -dimensional manifold, and suppose that there is a map $f: S^n \rightarrow M$ such that the induced homomorphism $f_*: H_n(S^n) \rightarrow H_n(M)$ is non-trivial. Compute $H_k(M; \mathbb{Q})$ for all k .
- (17) Prove that a map $f: \mathbb{C}\mathbb{P}^4 \rightarrow \mathbb{C}\mathbb{P}^2 \times \mathbb{C}\mathbb{P}^2$ must induce the trivial map on cohomology in all positive degrees.