

# The Stability of Walrasian General Equilibrium under a Replicator Dynamic

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## Abstract

We prove the stability of equilibrium in a completely decentralized Walrasian general equilibrium economy in which prices are fully controlled by economic agents, with production and trade occurring out of equilibrium.

Journal of Economic Literature Classifications:

C62—Existence and Stability Conditions of Equilibrium

D51—Exchange and Production Economies

D58—Computable and Other Applied General Equilibrium Economies

## 1 Introduction

Walras (1874) developed a general model of competitive market exchange, but provided only an informal argument for the existence of a market-clearing equilibrium for this model. Wald (1951) provided a proof of existence for a simplified version of Walras' model, and this proof was substantially generalized by Debreu (1952), Arrow and Debreu (1954), Gale (1955), Nikaido (1956), McKenzie (1959), Negishi (1960), and others.

Walras was well aware that his arguments had to be backed by a theory of price adjustment that would ensure stability of equilibrium. He considered the key force leading to equilibrium to be face-to-face competition, which he thought should result in the continual updating of prices by economic agents until equilibrium

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emerges. However, Walras believed that a model where economic agents individually update their prices would be analytically intractable, whereas a simple centralized model of price adjustment, the auctioneer playing the role of a representative agent, would more easily lend itself to a proof of the stability of equilibrium.

The stability of the Walrasian economy became a central research focus in the years following the existence proofs (Arrow and Hurwicz 1958, 1959, 1960; Arrow, Block and Hurwicz 1959; Nikaido 1959; McKenzie 1960; Nikaido and Uzawa 1960). Following Walras' tâtonnement process, these models assumed that there is no production or trade until equilibrium prices are attained, and out of equilibrium, there is a price profile shared by all agents, the time rate of change of which is a function of excess demand. These efforts at proving stability were successful only by assuming narrow and implausible conditions without microeconomic foundations (Fisher 1983). Indeed, Scarf (1960) and Gale (1963) provided simple examples of unstable Walrasian equilibria under a tâtonnement dynamic.

Several researchers then explored the possibility that allowing trading out of equilibrium could sharpen stability theorems (Uzawa 1959, 1961, 1962; Negishi 1961; Hahn 1962; Hahn and Negishi 1962, Fisher 1970, 1972, 1973), but these efforts did not provide micro-level conditions that ensure stability. Moreover, Sonnenschein (1973), Mantel (1974, 1976), and Debreu (1974) showed that any continuous function, homogeneous of degree zero in prices, and satisfying Walras' Law, is the excess demand function for some Walrasian economy. These results showed that no general stability theorem could be obtained based on the tâtonnement process. Indeed, subsequent analysis showed that chaos in price movements is the generic case for the tâtonnement adjustment processes (Saari 1985, Bala and Majumdar 1992).

A novel approach to the dynamics of large-scale social systems, evolutionary game theory, was initiated by Maynard Smith and Price (1973), and adapted to dynamical systems theory in subsequent years (Taylor and Jonker 1978, Friedman 1991, Weibull 1995). The application of these models to economics involved the shift from biological reproduction to behavioral imitation as the mechanism for the replication of successful agents.

A number of issues related to markets' dynamics have been investigated through this evolutionary lens. Vega-Redondo (1997) analyzes the convergence to the Walrasian outcome in a Cournot oligopoly where firms update quantities in an evolutionary fashion, Alos et al. (2000) provide an evolutionary model of Bertrand competition, Serrano and Volij (2008) study the stability of the Walrasian outcome in markets for indivisible goods, and a number of contributions (e.g Mandel and Botta 2009, Kim 2011) investigate evolutionary dynamics in specific exchange economies.

The present contribution targets the same connection between evolutionary dy-

namics and Walrasian behavior but at a broader scale. We provide evolutionary foundations to the Walrasian equilibrium in complex exchanges economies. We treat an exchange economy as the stage game of an evolutionary process in which each agent is endowed in each period with goods he must trade to obtain the various goods he consumes. There are no inter-period exchanges. An agent's trade strategy consists of a set of *private prices* for the goods he produces and the goods he consumes, such that, according to the individual's private prices, a trade is acceptable if the value of goods received is at least as great as the value of the goods offered in exchange. This representation of the exchange process has strong commonalities with the strategic bargaining literature, and in particular Rubinstein and Wolinsky (1985,1990), Gale (1986), Binmore and Herrero (1988), but the defining feature of our approach is to consider that agents use private prices as strategies.

This perspective on prices is consistent with the fact that prices are set in a decentralized manner in market economies and also consonant with Walras' (1874) description of the workings of competition in decentralized markets. In Walras' original description as well as in our setting, competition is effective because a seller who slightly undercuts his competitors increases his sales and hence his income or because a buyer who overbids his competitors increases his purchases and hence his utility. When markets are in disequilibrium, there are always agents who have an incentives to change their prices. In other words, under rather mild conditions on the trading process, Walrasian equilibria are the only strict Nash equilibria of our private prices games. Thus, if we assume that the strategies of traders are updated according to the replicator dynamic, the stability of equilibrium is guaranteed (Weibull 1995).

The use of private prices in the general equilibrium model results in a model closer to Walras' original insights where prices are controlled by economic agents as in actual markets, and allows us to address the stability problem using well-established evolutionary game-theoretic principles.

The paper is organized as follows: Section 2 introduces our model economy. Section 3 defines a class of bargaining games based on private prices in this economy for which we provide an example in section 4. Section 5 shows how learning dynamics in these games induce price dynamics in the underlying economy and reviews the stability properties of evolutionary dynamics. Section 6 proves the stability of equilibrium in a stylized setting where the kinds of goods consumed and sold by each agent are independent of relative prices and section 7 extends this result to an arbitrary exchange economy. Section 8 discusses in greater detail the necessary conditions for competition to entail a stable price adjustment process in a setting with private prices. Section 9 reviews the results of our analysis.

## 2 The Walrasian Economy

We consider an economy with a finite number of goods, indexed by  $h = 1, \dots, l$ , and a finite number of agents, indexed by  $i = 1, \dots, m$ . Agent  $i$  has  $\mathbf{R}_+^l$  as consumption set, a utility function  $u_i: \mathbf{R}_+^l \rightarrow \mathbf{R}_+$  and an initial endowment  $e_i \in \mathbf{R}_+^l$ . We shall denote this economy by  $\mathcal{E}(u, e)$  and consider the following standard concepts for its analysis:

- An allocation  $\bar{x} \in (\mathbf{R}_+^l)^m$  of goods is *feasible* if it belongs to the set  $\mathcal{A}(e) = \{x \in (\mathbf{R}_+^l)^m \mid \sum_{i=1}^m x_i \leq \sum_{i=1}^m e_i\}$ .
- The demand of an agent  $i$  is the mapping  $d_i: \mathbf{R}_+^l \times \mathbf{R}_+ \rightarrow \mathbf{R}_+^l$  that associates to a price  $p \in \mathbf{R}_+^l$  and an income  $w \in \mathbf{R}_+$ , the utility-maximizing individual allocations satisfying the budget constraint

$$d_i(p, w) = \operatorname{argmax}\{u(x_i) \mid x_i \in \mathbf{R}_+^l, p \cdot x_i \leq w\}.$$

- A feasible allocation  $\bar{x} \in \mathcal{A}(e)$  is an *equilibrium allocation* if there exists a price  $\bar{p} \in \mathbf{R}_+^l$  such that for all  $i$ ,  $\bar{x}_i \in d_i(\bar{p}, \bar{p} \cdot e_i)$ . The price  $\bar{p}$  is then called an *equilibrium price*. We shall denote the set of such equilibrium prices by  $\mathbf{E}(u, e)$ .
- Additionally, a feasible allocation  $x^* \in \mathcal{A}(e)$  and a price  $p^* \in \mathbf{R}_+^l$  form a *quasi-equilibrium* if for all  $x_i \in \mathbf{R}_+^l$ ,  $u_i(x_i) > u_i(x_i^*)$  implies  $p^* \cdot x_i > p^* \cdot x_i^*$ .

Our focus is on the stability of equilibrium. Of course, this presupposes there actually are equilibria in the economy. Therefore we place ourselves throughout the paper in a setting where sufficient conditions for the existence of an equilibrium hold.

First, in order to ensure the existence of a quasi-equilibrium (Florenzano 2005), we assume that utility functions satisfy the following standard assumptions (the strict concavity is not necessary for the existence of a quasi-equilibrium but implies demand mappings are single-valued, which will prove useful below).

**Assumption 1 (Utility)** *For all  $i = 1, \dots, m$ ,  $u_i$  is continuous, strictly concave, and locally non-satiated.*

Second, to ensure that every quasi-equilibrium is an equilibrium allocation, it suffices to assume that at a quasi-equilibrium the agents do not receive the minimal possible income (Hammond 1993, Florenzano 2005). This condition is satisfied under the survival assumption (i.e when all initial endowments are in the interior

of the consumption set) as well as in settings with corner endowments such as those investigated in Scarf (1960) and Gintis (2007). Formally, the assumption can be stated as follows.

**Assumption 2 (Income)** *For every quasi-equilibrium  $(p^*, x^*)$ , and for every  $i = 1, \dots, m$ , there exists  $x_i \in \mathbf{R}_+^l$  such that  $p^* \cdot x_i^* > p^* \cdot x_i$ .*

It is standard to show that under Assumptions (1) and (2) the economy  $\mathcal{E}(u, e)$  has at least an equilibrium (Florenzano 2005). Hence, we shall assume in the following they do hold. Moreover, we shall restrict attention to the generic case where the economy has a finite set of equilibria (Balasko 2009).

### 3 Exchange Processes with Private Prices

In his *Elements of Pure Economics* (1874), Walras envisages equilibrium emerging as the outcome of free competition among economic agents. He characterizes free competition as the combination of (i) the free entry and exit in the market, (ii) the capacity of producers to choose their level of production and, (iii) the freedom of traders to set and modify their prices (Dockès and Potier 2005). It is striking Walras emphasizes the private nature of prices—note his use of possessive articles below—set in a decentralized manner by the traders themselves. Walras writes:

As buyers, traders make their demand by outbidding each other. As sellers, traders make their offers by underbidding each other. . . The markets that are best organized from the competitive standpoint are those in which. . . the terms of every exchange are openly announced and an opportunity is given to sellers to lower their prices and to buyers to raise their bids (Walras 1984, paragraph 41).

Though he saw decentralized competition as the driving force towards equilibrium, it seems Walras considered that an aggregate representation of price adjustment based on the centralized tâtonnement process would provide a simpler model in which one could prove the convergence towards equilibrium and its stability. Modern work on the question unfortunately showed that the tâtonnement process lacked those desirable features.

We propose an alternative model of price adjustment based on individual learning by traders who both set prices and trade at these prices out-of-equilibrium. Treating traders as price-setters out of equilibrium is consonant both with Walras' description of free competition and the way actual trading takes place. Nor is this assumption at odds with the law of one price or the notion that economic agents are

price-takers under perfect competition, for both of these conditions hold in market equilibrium, and hence are long-run properties of our model.

The main building block of our approach is a game-theoretic representation of exchange processes based on private prices. Namely, we consider that each agent  $i$  in the economy is characterized by a private price  $p_i \in \mathbf{R}_+^l$  whose coordinates represent the prices at which he is willing to sell the goods he supplies to the market and the maximum prices he is willing to pay for the goods he demands. The allocation of goods in the economy is then determined by the distribution of private prices as well as by the initial endowments. In other words, we represent the exchange process as a game where agents use private prices as strategies. Formally, we shall associate to the economy  $\mathcal{E}(u, e)$  the class of games  $\mathcal{G}(u, e, \xi)$  where:

- Each agent has a finite set of prices  $P \subset \mathbf{R}_+^l$  as strategy set.
- The game form is defined by an exchange mechanism  $\xi: P^m \rightarrow \mathcal{A}(e)$  that associates to a profile of private prices  $\pi \in P^m$  an attainable allocation  $\xi(\pi) = (\xi_1(\pi), \dots, \xi_m(\pi)) \in \mathcal{A}(e)$ .
- The payoff  $\phi_i: P^m \rightarrow \mathbf{R}_+$  of player  $i$  is evaluated according to the utility of the allocation it receives, that is  $\phi_i(\pi) = u_i(\xi_i(\pi))$ .

A wide range of exchange processes can be embedded in this framework: processes based on a central clearing system such as double auctions, processes based on simultaneous and multilateral exchanges as usually considered in general equilibrium models with out-of-equilibrium features (Grandmont 1977, Benassy 2005), or processes akin to the tâtonnement where no trade takes place unless excess demand vanishes<sup>1</sup> (that is  $\xi_i(\pi) = d_i(\bar{p}, \bar{p} \cdot e_i)$  if for all  $i$   $\pi_i = \bar{p} \in \mathbf{E}(u, e)$  and  $\xi_i(\pi) = e_i$  otherwise). Yet, the most interesting examples from our point of view are processes based on sequences of bilateral trades such as those considered by Gintis (2007,2012) in Scarf and Leontief economies. These processes have formal commonalities, as far as the exchange process is concerned, with the bargaining models considered in Rubinstein and Wolinsky (1985), Gale (1987), or Binmore and Herrero (1988). Still, whereas these authors consider perfectly rational agents of two different types, we focus on large populations of rational agents with limited information about the general economy and who update their strategies in an evolutionary fashion. The following example illustrates, following Gintis (2007), how such a process can induce converging price dynamics in the Scarf economy, which is a well-known example of instability of the tâtonnement process.

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<sup>1</sup>This tâtonnement like process is a trivial example of exchange process satisfying the conditions put forward below.

#### 4 A simulation in the Scarf economy

We consider the Scarf economy with three goods ( $\ell = 3$ ) and three types of agents ( $m = 3$ ). Agent 1 has utility  $u_1(x_1, x_2, x_3) = \min(\frac{x_1}{\omega_1}, \frac{x_2}{\omega_2})$ , and initial endowment  $(\omega_1, 0, 0)$ , agent 2 has utility  $u_2(x_1, x_2, x_3) = \min(\frac{x_2}{\omega_2}, \frac{x_3}{\omega_3})$  and initial endowment  $(0, \omega_2, 0)$  and agent 3 has utility  $u_3(x_1, x_2, x_3) = \min(\frac{x_3}{\omega_3}, \frac{x_1}{\omega_1})$  and initial endowment  $(0, 0, \omega_3)$ . One sets  $(\omega_1, \omega_2, \omega_3) = (10, 20, 400)$  so that equilibrium relative prices are extremely unequal (see Anderson and al. 2004). Namely, the equilibrium price is proportional to  $(40, 20, 1)$ .

We then consider 1000 agents of every type that we initially endow with a set of private prices randomly drawn from the uniform distribution on  $[0, 1]$  and that we let evolve according to the following dynamics, which we implement numerically.

At the beginning of each period, the inventory of each agent is reset to the value of its initial endowment. Each agent in turn is then designated a trade initiator and is paired with a randomly chosen responder, who can either accept or reject the proposed trade. Each agent is thus an initiator exactly once and responder on average once per period.

The trade procedure is as follows. The initiator offers a certain quantity of one good in exchange for a certain quantity of a second good. If the responder has some of the second good, and if the value of what he gets exceeds the value of what he gives up, according to his private prices, then he agrees to trade. If he has less of the second good than the initiator wants, the trade is scaled down proportionally.

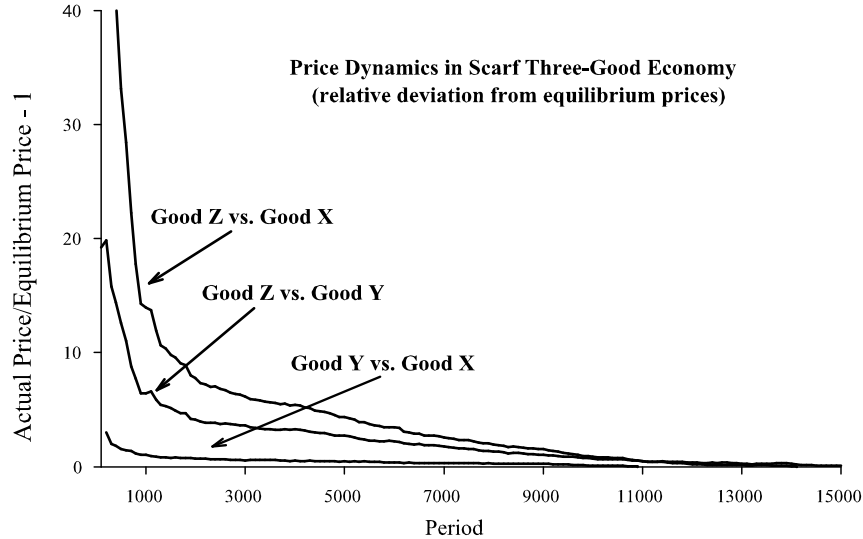
The initiators trade ratios are given by his private prices. Which good he offers to trade for which other good is determined as follows. Let us call an agents endowment good his P-good, the additional good he consumes his C-good, and the good which he neither produces nor consumes his T-good. If the initiator has his T-good in inventory, he offers to trade this for his C-good. If this offer is rejected, he offers to trade for his P-good. If the initiator does not have his T-good but has his P-good, he offers this in trade for his C-good. If this is rejected, he offers to trade half his P-good for his T-good. If the initiator had neither his T-good nor his P-good, he offers his C-good in trade for his P-good, and if this fails he offers to trade for his T-good. In all cases, when a trade is carried out, the terms are dictated by the initiator and the amount is the maximum compatible with the inventories of the initiator and responder. After a successful trade, agents consume whatever is feasible from their updated inventory.

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must be willing to acquire their T-good despite the fact that it does not enter their utility function. This is because X-producers want Y, but Y-producers do not want X. Only Z-producers want X. Since a similar situation holds with Y-producers and Z-producers, consumption ultimately depends on at least one type of producer accepting the T-good in trade, and then using the T-good to purchase their C-good. C-good. If this offer is rejected, he offers to trade for his P-good. This may sound bizarre, but it conforms to the general rule of agreeing to trade as long as the value of ones inventory increases. If the initiator does not have his T-good but has his P-good, he offers this in trade for his C-good. If this is rejected, he offers to trade half his P-good for his T-good. If the initiator had neither his T-good nor his P-good, he offers his C -good in trade for his P-good, and if this fails he offers to trade for his T-good. In all cases, when a trade is carried out, the terms are dictated by the initiator and the amount is the maximum compatible with the inventories of the initiator and responder. After a successful trade, agents consume whatever is feasible from their updated inventory.

Then, every ten periods occurs a reproduction stage in which 5% of agents are randomly chosen either to copy a more successful agent or to be copied by a less successful agent, where success is measured by total undiscounted utility of consumption over the previous ten periods. Such an agent is chosen randomly and assigned a randomly chosen partner with the same production and consumption parameters. The less successful of the pair then copies the private prices of the more successful. In addition, after the reproduction stage, each price of each agent is mutated with 1% probability, the new price either increasing or decreasing by 10%.





**Figure 1:** Convergence of Price to Equilibrium in a Three-good Scarf Economy with Private Prices

The results of a typical run are exhibited in figure 1. Each of the curves in Figure 3 is given by  $\frac{p^* - p}{p^*}$  where  $p^*$  is the equilibrium relative price. Because initial prices are generated by uniform distributions on the unit interval, the initial mean price of good 3 is 40 times its equilibrium value, and the price of good 2 is 20 times its equilibrium value. Nevertheless, in sharp contrast to the oscillatory behavior of the tâtonnement process in this setting (see Scarf 1960), the system settles rapidly in a steady state where prices assume their equilibrium values, i.e we observe convergence to equilibrium.

Similar convergence results have been obtained in economies with Leontieff preferences (see Gintis 2012) as well as in arbitrarily large economies with nested C.E.S utility functions where fitness is measured by traded volumes rather than by utility (Gintis 2013). This set of results suggest that in a setting where agents use private prices as strategies, evolutionary dynamics based on “Aspiration and imitation of success” (see Schlag, 1998 and Benaim and Weibull, 2003) can provide dynamical foundations for general equilibrium in settings where the tâtonnement fails.

## 5 Learning and Price Dynamics

In general, if individual private prices are recognized as strategic variables, price dynamics shall depend on the way agents react to the utility these prices yield. In a setting that is supposed to represent a competitive economy and hence to encompass a very large number of agents, it seems illusory to consider that any single agent has the information processing and the computational abilities to choose the best response to the complete profile of prices. Assuming rational agents with limited information who learn through their own experience and that of others is a superior model, provided the resources and the technology of the economy (the initial endowments in our setting) evolve slowly enough with respect to the frequency of transactions, that past experience is indicative of future conditions.

The price dynamics described in the preceding section constitute a learning process. We construct a multi-population game by assuming there is a large population of agents of each type (a type being characterized here by an initial endowment and a utility function). Second, a series of private-price games are played every period. Third, a number of agents update their strategies in parallel by imitating agents that performed better than themselves in the private-price games. The resulting price adjustment process is a Markov process whose global stability properties can be analyzed using the notion of stochastic stability (see Kandori and al. 1993, Peyton-Young 1993). In previous contributions (Mandel and Botta 2009, Gintis and Mandel 2014), we have shown that in the Scarf and Leontief economy investigated numerically in Gintis (2007,2013), the Walrasian equilibrium is the only stochastically stable state for a large class of private-price games and hence obtained a global convergence result. Yet, stochastic stability methods require a detailed understanding of the payoff structure. Hence such methods are well suited for analyzing the stability of equilibrium in specific models such as this of Vega-Redondo (1997) but less so when it comes to obtain stability results in an arbitrary exchange economies in which the number of types and the exact functional forms of utility functions are left unspecified. In the latter case with which we are concerned in this paper, the use of more powerful learning models, such as the replicator dynamics, whose generic properties are well understood seems more appropriate. In this case, we can associate a population model to a private price game of the kind introduced in section 3 by considering that for each  $i = 1 \dots m$ , there is a set of agents with utility function  $u_i$  and initial endowment  $e_i$ . Those agents form the population of type  $i$ . The distribution of prices/strategies in a population of type  $i$  can then be identified with a mixed strategy in the set

$$\Delta_i = \left\{ \sigma_i \in \mathbf{R}_+^P \mid \sum_{p \in P} \sigma_{i,p} = 1 \right\}$$

where  $\sigma_{i,p}$  is the share of agents in population  $i$  using price  $p \in P$ .

Imitation models giving rise to a replicator dynamic can be introduced in this setting by considering that at sequential times, one individual in one population is drawn at random to update his price/strategy. In the model of “aspiration and random imitation” (Gale, Binmore and Samuelson 1995, Björnerstedt and Weibull 1996 and Binmore and Samuelson 1997), the reviewing agent switches to the strategy of a randomly chosen agent of his population if the expected payoff of his current strategy is below some aspiration level. In the model of “proportional imitation model” of Schlag (1998), the reviewing agent observes a random individual in his population and switches to the latter’s strategy with a probability proportional to the payoff difference. Both models can be seen as versions of the imitation models introduced in the numerical example of the preceding section. In this respect, one should note that the implementation of imitation in our setting does not require that agents observe each others’ utilities: an agent identifies a peer of his population through his initial endowment and he only needs to observe the peer’s consumption in order to compare their respective utilities. As a matter of fact, in Gintis (2013), fitness is measured by traded volumes rather than by utility.

It is well-known (see e.g Benaim and Weibull 2003) that in the large-population limit, the two models of imitation introduced above can be approximated by the replicator dynamic, which assumes that the share of agents using a given strategy grows proportionally to the utility it is expected under the assumption that opponent players are drawn uniformly in each of the populations. As a matter of fact, although there is a number of competing models of individual learning in games, from fictitious play to reinforcement learning through bayesian updating or stochastic imitation (Fudenberg and Levine 1997), the replicator equation forms the backbone of most of these approaches: (Hopkins 2002) shows that the expected motion of stochastic fictitious play and reinforcement learning with experimentation can both be written as a perturbed form of the evolutionary replicator dynamics, Helbing (1996) proves a similar result for stochastic imitation models while Shalizi (2009) highlights the rather mild conditions under which Bayesian updating and the replicator dynamics are equivalent.

Taking stock of this literature, we focus in the following on the stability properties of general equilibrium in private price games where learning takes place according to the replicator dynamics. In this setting, the replicator dynamic prescribes that the share of agents using price  $p$  in the population  $i$  grow proportionally to the utility it is expected to yield in the exchange process under the assumption that trade partners are drawn uniformly in each of the populations. This yields the system of differential equations defined for all  $i = 1, \dots, m$  and  $p \in P$  by:

$$\frac{\partial \sigma_{i,p}}{\partial t} = \sigma_{i,p} (E_{\sigma_{-i}}(u_i(\xi(p, \cdot))) - E_{\sigma}(u_i(\xi(\cdot)))) \quad (1)$$

where  $E_{\sigma_{-i}}(u_i(\xi(p, \cdot)))$  represents the expected utility of the strategy  $p$  given the mixed strategy profile  $\sigma_{-i}$ , that is

$$E_{\sigma_{-i}}(u_i(\xi(p, \cdot))) = \sum_{\rho \in P^{m-1}} \left( \prod_{j \neq i} \sigma_{j, \rho_j} \right) u_i(\xi(p, \rho)),$$

and  $E_{\sigma}(u_i(\xi(\cdot)))$  represents the expected utility of the mixed strategy  $\sigma_i$  given the mixed strategy profile  $\sigma_{-i}$ ; that is

$$E_{\sigma}(u_i(\xi(\cdot))) = \sum_{\pi \in P^m} \left( \prod_{j=1}^m \sigma_{j, \pi_j} \right) u_i(\xi(\pi)).$$

In multi-population games such as  $\mathcal{G}(u, e, \xi)$ , the replicator dynamic has very salient equilibrium selection properties: a strategy profile is (locally) asymptotically stable for the replicator dynamics if and only if it is a strict Nash equilibrium of the underlying game<sup>2</sup> (see Weibull 1995 and Appendix A below). In other words, one has:

**Proposition 1** *A price profile  $\pi \in P^m$  is asymptotically stable for the replicator dynamic if and only if for all  $i = 1 \dots m$ , and all  $p \neq \pi_i$ , one has  $u_i(\pi) > u_i(p, \pi_{-i})$ .*

Therefore, to prove the asymptotic stability of the economic equilibria of the economy  $\mathcal{E}(u, e)$ , it suffices to show that these equilibria can be identified with strict Nash equilibria of the game  $\mathcal{G}(u, e, \xi)$ . This is the purpose of the remaining of this paper. Note that on top of asymptotic stability, this would guarantee that general economic equilibria are both risk-dominant (as the only strict equilibria) and Pareto-dominant (given the first welfare theorem).

## 6 An Axiomatic Characterization of Stable Exchange Processes

From this point forward, we consider as price set  $P = K^{l-1} \times \{1\}$ , where  $K \subset \mathbf{R}_+$  is a finite set of commodity prices with minimum  $p_{\min} > 0$  and maximum  $p_{\max} > p_{\min}$  while good  $l$  is used as a numeraire and its price is fixed equal to 1. We also consider that the price set  $P$  contains each of the finite number of equilibrium prices of the economy.

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<sup>2</sup>Similar results for a broader class of dynamics follow from the application of the results recalled in Appendix A.

As underlined in section 3, a wide range of exchange processes can be represented as games of the form  $\mathcal{G}(u, e, \xi)$ . On the one hand, it is clear that equilibrium stability can't hold for every exchange process  $\xi$ , on the other hand a stability result that would hold for a single process won't do justice to the variety of market structures that can be deemed competitive. In order to avoid both pitfalls, we adopt an axiomatic approach and characterize a large class of exchange processes  $\xi$ , such that the only strict Nash equilibria of the game  $\mathcal{G}(u, e, \xi)$  are strategy profiles  $\bar{\pi}$  such that each agent uses the same general equilibrium price, that is there exists  $\bar{p} \in \mathbf{E}(u, e)$  such that for all  $i \in \{1, \dots, m\}$ ,  $\pi_i = \bar{p}$ . In other words, we give sufficient conditions for the general equilibria of the economy  $\mathcal{E}(u, e)$  to be the only asymptotically stable states of the replicator dynamics in the game  $\mathcal{G}(u, e, \xi)$ .

It turns out that the analysis is simpler in a setting where for each agent the set of goods is partitioned between consumption goods (those the agent consumes) and production goods (those the agent is endowed with). We first focus on this particular case and treat the general case in the next section. Hence, we assume within this section that the following assumption holds.

**Assumption 3 (Goods partition)** *For all  $i = 1, \dots, m$ , there exists a partition  $\{P_i, C_i\}$  of  $\{1, \dots, l\}$  such that:*

1. *for all  $h \in P_i$ ,  $e_{i,h} > 0$ ;*
2. *there exists  $v_i : V_{C_i} \rightarrow \mathbf{R}_+^3$  such that for all  $x \in \mathbf{R}_+^l$ :*
  - (a)  $u_i(x) = v_i(\text{pr}_{V_{C_i}}(x))$
  - (b)  $v_i(x) > 0 \Rightarrow \forall h \in C_i, x_h > 0$ .

That is  $P_i$  are the goods produced/sold by agent  $i$  and  $C_i$  those he consumes/buys. Accordingly, we define the set of buyers of good  $h$  as  $B_h := \{i \mid h \in C_i\}$ , and the set of sellers of good  $h$  as  $S_h := \{i \mid h \in P_i\}$ .

Now the core of our approach to prove stability as follows : general equilibria will be stable if agents have incentives (i) to agree on a common price in order to increase the size of the market and the number of opportunities for Pareto improving trades and (ii) to deviate from the common price by underbidding or overbidding their competitors when there is excess supply or excess demand.

Incentives to agree on a common price are related to the constraints private prices put on trading possibilities. In this respect, the less stringent condition on

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<sup>3</sup>For a subset  $C \subset \{1, \dots, l\}$ , we define the vector subspace  $V_C$  as  $V_C := \{x \in \mathbf{R}^l \mid \forall h \notin C, x_h = 0\}$  and  $\text{pr}_{V_C}$  as the projection on  $V_C$ .

access to the market we can assume is to consider that access is restricted to traders that have a possibility to find a trade partner: buyers whose private price for a good is above the lowest private price of sellers, and conversely sellers whose private price for a good is below the highest private price of buyers (see respectively conditions 1 and 2 in definition 1 below). This subsumes more stringent conditions, such as buying only from sellers using a price below one's own (such a condition can be expressed thanks to a detailed description of a particular exchange process, but not at the level of generality at which we place ourselves).

We must also account for the budgetary constraint an agent's private price put on his allocation by assuming an agent's allocation is less than its utility maximizing allocation for his private prices (see condition 3 in definition 1 below). This latter condition is also in line with the fact that, out of equilibrium, there is no public price with regards to which agents can act as price takers: they must evaluate goods according to their private prices and determine their behavior accordingly.

Formally, given a private price profile  $\pi$ , we define the set of acceptable buyers as those whose prices is above the lowest buying price, that is  $\mathcal{B}_h(\pi) := \{i \in B_h \mid \pi_{i,h} \geq \min_{j \in S_h} \pi_{j,h}\}$ . the set of acceptable sellers as those whose prices is below the highest buying price, that is  $\mathcal{S}_h(\pi) := \{i \in S_h \mid \pi_{j,h} \leq \max_{i \in B_h} \pi_{i,h}\}$  and the feasible income as  $w_i(\pi) = \sum_{\{h \mid i \in \mathcal{S}_h(\pi)\}} \pi_{i,h} e_h$ . The set of price feasible allocations  $\mathcal{A}'(e, \pi)$  is then defined as follows.

**Definition 1** *The set of price feasible allocations  $\mathcal{A}'(e, \pi)$  is the subset of feasible allocations  $\mathcal{A}(e)$  such that for every  $x \in \mathcal{A}'(e, \pi)$  one has:*

1.  $\forall h \in C_i, x_{i,h} > 0 \Rightarrow i \in \mathcal{B}_h(\pi)$ ;
2.  $\sum_{i \in \mathcal{B}_h(\pi)} x_{i,h} \leq \sum_{j \in \mathcal{S}_h(\pi)} e_{j,h}$ ;
3.  $x_i \leq d_i(\pi_i, w_i(\pi))$ .

Following the above discussion, condition 1 states that only buyers with acceptable prices have access to the market. Conversely, condition 2 states that the relevant supply is this from agents who have access to the market. Condition 3 expresses the fact that demand are computed at private prices<sup>4</sup>.

From a broader point of view, definition 1 expresses the fact that as agents agree on the valuation of goods, i.e as private prices become more similar, the size of the market grows. In particular when a uniform price profile is reached, i.e when  $\pi \in P^m$  is such that there exists  $p \in P$  such that for all  $i \in \{1, \dots, m\}$   $\pi_i = p$ ,

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<sup>4</sup>it would be equivalent for our purposes to assume that each agent satisfies its private budget constraint, that is  $\pi_i \cdot x_i \leq w_i(\pi)$ .

then the market includes each agent, and the constraint his private price put on a given agent is simply his private budget constraint.

Now, agents will have incentives to agree on a common price if they actually gain from an increased size of the market. To express mathematically this idea, it is useful to see the set of price feasible allocations  $\mathcal{A}'(e, \pi)$  as a bargaining set defined by the agents' private prices. Then, the fact that an agent gains from an increase in the market size (provided his budget constraint is not being strengthened) simply appears as the transcription in our context of the monotonicity condition that is standard in the bargaining literature since the seminal paper by Kalai and Smorodinsky (1975)

**Assumption 4 (Monotonicity)** *If  $\pi$  and  $\pi'$  are such that  $\mathcal{A}'(e, \pi) \subset \mathcal{A}'(e, \pi')$  then for all  $i = 1 \dots n$ , one has  $u_i(\xi_i(\pi')) \geq u_i(\xi_i(\pi))$ .*

Assumption 4 will guarantee that agents have incentives to reach a uniform price profile. If this price is an equilibrium price, that is if  $\bar{\pi} \in P^m$  is such that for all  $i \in \{1, \dots, m\}$ ,  $\bar{\pi}_i = \bar{p}$  where  $\bar{p} \in \mathbf{E}(u, e)$ , we shall call  $\bar{\pi}$  an equilibrium price profile. Our approach to the issue of stability is then based on the identification of equilibrium price profiles of  $\mathcal{G}(u, e, \xi)$  with equilibria of the economy  $\mathcal{E}(u, e)$ . That is we shall assume that at an equilibrium price profile, the corresponding equilibrium allocation prevails.

**Assumption 5 (Equilibrium)** *If  $\bar{\pi} \in P^m$  is such that for all  $i \in \{1, \dots, m\}$ ,  $\bar{\pi}_i = \bar{p}$  where  $\bar{p} \in \mathbf{E}(u, e)$ , then one has for all  $i \in \{1, \dots, m\}$ :*

$$\xi_i(\bar{\pi}) = d_i(\bar{p}, \bar{p} \cdot e_i).$$

Note that this is in fact a very weak efficiency requirement on the exchange process  $\xi$ . Indeed, condition (3) in the definition of  $\mathcal{A}'(\pi, e)$  implies that for all  $i \in \{1, \dots, m\}$ ,  $\xi_i(\bar{\pi}) \leq d_i(\bar{p}, \bar{p} \cdot e_i)$ . As moreover  $(d_i(\bar{p}, \bar{p} \cdot e_i))_{i=1, \dots, m}$  indeed is a price feasible allocation, assuming that for all  $i \in \{1, \dots, m\}$ , one has  $\xi_i(\bar{\pi}) = d_i(\bar{p}, \bar{p} \cdot e_i)$ , in fact amounts to consider that the exchange process is efficient (in the sense that it picks up a Pareto optimal allocation) at equilibrium price profiles. This is much weaker than what is usually assumed of bargaining solutions, which are thought to be efficient for any given of the bargaining problem.

Eventually, emergence and stability of equilibrium will depend on the effectiveness of competition in the economy. At a non-equilibrium uniform price profile, that is if  $\pi \in P^m$  is such that for all  $i \in \{1, \dots, m\}$ ,  $\pi_i = p$  where  $p \notin \mathbf{E}(u, e)$ , Walras' law implies that there necessarily is excess demand for some good. We shall restrict attention to exchange processes that are competitive in the sense that in case of excess demand, at least one agent has incentives to deviate from the common price. This is the counterpart of Walras' description of buyers raising their bids to price out competitors. Namely, we shall assume that:

**Assumption 6 (Competition)** *If  $\pi = (p, \dots, p) \in P^m$  is a uniform price profile and there exists an  $h \in \{1, \dots, l\}$  such that*

$$\sum_{i=1}^m d_{i,h}(p, w_i(\pi)) > \sum_{i=1}^m e_{i,h},$$

*then there exists  $i \in \{1, \dots, m\}$  and  $p' \in P$  such that  $u_i(\xi_i(p', \pi_{-i})) \geq u_i(\xi_i(\pi))$ .*

An extensive discussion of the conditions under which exchange processes actually are competitive in the sense of this assumption is developed in section 8. Basically, an exchange process will be competitive if it implements some form of priority for lowest bidding sellers and highest bidding buyers. It is then clear that when there is excess supply a seller who slightly undercuts his competitors increases his sales and hence his income while when there is excess demand, a buyer who slightly overbids his competitors overcomes rationing. Note that in our framework, it does not make sense for a seller to increase his price before buyers do as he would price himself out of the market.

Yet, out of equilibrium, there always are agents who have incentives to deviate. Conversely, at equilibrium, an agent who deviates either prices himself out of the market or decreases his consumption and hence his utility. Hence, only Walrasian equilibria can be strict/stable Nash equilibria of the private price game. A detailed proof is given below.

**Proposition 2** *Under Assumptions (3) – (6),  $\pi \in \Pi$  is a strict Nash equilibrium of  $\mathcal{G}(u, e, \xi)$  if and only if  $\bar{\pi}$  is an equilibrium price profile.*

**Proof:** *According to assumption 5, if  $\bar{\pi} = (\bar{p}, \dots, \bar{p})$  is an equilibrium price profile then for all  $i$ ,  $\xi_i(\bar{\pi}) = d_i(\bar{p}, w_i(\bar{\pi}))$ , where  $w_i(\bar{\pi}) = \bar{p} \cdot e_i$ . Assume then agent  $i$  deviates to a price  $p \neq \bar{p}$ . As for his production goods  $h \in P_i$ , one has:*

- *If  $p_h > \bar{p}_h$ , one has  $i \notin S_h(p', \bar{\pi}_{-i})$  so that agent  $i$  can no longer sale good  $h$  and does not rise any income on the good  $h$  market.*
- *If  $p_h < \bar{p}_h$ , one has  $p_h e_{i,h} < \bar{p}_h e_{i,h}$ .*
- *If  $p_h = \bar{p}_h$ , one has  $p_h e_{i,h} = \bar{p}_h e_{i,h}$*

*Hence, either one has for all  $h \in P_i$ ,  $p_h = \bar{p}_h$ , and  $w_i(p', \bar{\pi}_{-i}) = w_i(\bar{\pi})$ , either one has  $w_i(p', \bar{\pi}_{-i}) < w_i(\bar{\pi})$ .*

*Looking then at consumption goods, one has:*



- Either there exists  $h \in C_i$  such that  $p_h < \bar{p}_h$ , so that  $i \notin \mathcal{B}_h(p', \bar{\pi}_{-i})$  and  $\xi_{i,h}(p', \bar{\pi}_{-i}) = 0$ , which implies according to assumption 3.2b that  $u_i(\xi_{i,h}(p', \bar{\pi}_{-i})) = 0 < u_i(d_i(\bar{p}, \bar{p} \cdot e_i))$ .
- Either one has for all  $h \in C_i$ ,  $p_h \geq \bar{p}_h$ , with the inequality being strict for some  $h$ . As according to the preceding, one always has  $w_i(p', \bar{\pi}_{-i}) \leq w_i(\bar{\pi})$ , it must be that  $u_i(d_i(p', w_i(p', \bar{\pi}_{-i}))) < u_i(d_i(\bar{p}, w_i(\bar{\pi})))$  so that  $u_i(\xi_i(p', \bar{\pi}_{-i})) \leq u_i(d_i(p', w_i(p', \bar{\pi}_{-i}))) < u_i(d_i(\bar{p}, w_i(\bar{\pi}))) = u_i(\xi_i(\bar{\pi}))$ .
- Either for all  $h \in C_i$ ,  $p_h = \bar{p}_h$ , so that one necessarily has  $w_i(p', \bar{\pi}_{-i}) < w_i(\bar{\pi})$  according to the preceding. Again, one has  $u_i(d_i(p', w_i(p', \bar{\pi}_{-i}))) < u_i(d_i(\bar{p}, w_i(\bar{\pi})))$  and therefore  $u_i(\xi_i(p', \bar{\pi}_{-i})) < u_i(\xi_i(\bar{\pi}))$ .

To sum up, if agent  $i$  deviates to price  $p'$  either his income or his choice set are strictly reduced: he cannot by definition obtain an allocation as good as  $d_i(\bar{p}, w_i(\bar{\pi}))$  and hence is strictly worse off.

Suppose then  $\pi$  is not an equilibrium price profile.

- If  $\pi$  is a uniform price profile and there exists an  $h \in \{1, \dots, l\}$  such that  $\sum_{i=1}^m d_{i,h}(p, w_i(\pi)) > \sum_{i=1}^m e_{i,h}$ , then assumption 6 implies  $\pi$  is not a (strict) Nash equilibrium.

Hence, from here on we can assume that  $\pi$  is not a uniform price profile.

- If there exists  $h \in \{1, \dots, \ell\}$  and  $i \in S_h / \mathcal{S}_h(\pi)$ , by setting  $p'_h := \max_{i \in B_h} \pi_{i,h}$  and  $p'_j := \pi_{i,j}$  for all  $j \neq h$  one obtains a population  $\pi' := (p', \pi_{-i})$  such that  $w_i(\pi') \geq w_i(\pi)$  while the other constraints in the definition of  $\mathcal{A}'$  can only be relaxed. Therefore,  $\mathcal{A}'(e, \pi) \subset \mathcal{A}'(e, \pi')$ . According to assumption 4, it can not be that  $\pi$  is a strict Nash equilibrium. Similar arguments apply whenever there exists  $h \in \{1, \dots, \ell\}$  and  $i \in B_h / \mathcal{B}_h(\pi)$ .
- Otherwise, one must have  $B_h = \mathcal{B}_h(\pi)$ ,  $S_h = \mathcal{S}_h(\pi)$ , and there must exist  $i, i' \in \{1, \dots, m\}$  and  $h \in \{1, \dots, l\}$  such that  $\pi_{i,h} \neq \pi_{i',h}$ , e.g.  $\pi_{i,h} < \pi_{i',h}$ . One then has:
  - If  $i, i' \in S_h$  by setting  $\pi'_{i,h} = \pi_{i',h}$  and  $\pi'_{j,k} = \pi_{j,k}$  for  $j \neq i'$  or  $k \neq h$ , one has  $w_i(\pi') \geq w_i(\pi)$  and hence  $\mathcal{A}'(e, \pi) \subset \mathcal{A}'(e, \pi')$  so that  $\pi$  can not be a strict Nash equilibrium according to assumption 4. The same holds true if  $i \in S_h$  and  $i' \in B_h$ .

Hence, from here on we can assume that for all  $i, i' \in S_h$ ,  $\pi'_{i,h} = \pi_{i',h}$ . Then:

- If  $i, i' \in B_h$  by setting  $\pi'_{i',h} = \pi_{i,h}$  and  $\pi'_{j,k} = \pi_{j,k}$  for  $j \neq i$  or  $k \neq h$ , the private budget constraint of agent  $i'$  (condition 3 in definition of  $\mathcal{A}'$ ) is relaxed and one then has  $\mathcal{A}'(e, \pi) \subset \mathcal{A}'(e, \pi')$  so that  $\pi$  can not be a strict Nash equilibrium according to assumption 4.
- If  $i \in B_h$  and  $i' \in S_h$ , it must be, given that  $S_h = S_h(\pi)$ , that there exist  $i'' \in B_h$  such that  $\pi_{i',h} \leq \pi_{i'',h}$  and the preceding argument applies to  $i, i'' \in B_h$  such that  $\pi_{i,h} < \pi_{i'',h}$ .

Summing up,  $\pi$  cannot be a strict Nash equilibrium of  $\mathcal{G}(u, e, \xi)$  if it is not an equilibrium price profile in the Walrasian sense. This ends the proof.

Through proposition 1, a direct corollary of proposition 2 is the asymptotic stability of equilibrium for the replicator dynamics.

**Proposition 3** *Under assumptions (3) through (6), the only asymptotically stable strategy profiles for the replicator dynamic in  $\mathcal{G}(u, e, \xi)$  are those for which each agent uses an equilibrium price  $\bar{p} \in \mathcal{E}(u, e)$  and agent  $i$  is allocated his equilibrium allocation  $d_i(\bar{p}, \bar{p} \cdot e_i)$ .*

This result strongly contrasts with the lack of generic stability of the tâtonnement. Both processes are driven by competition, the adaptation of prices to market condition. Yet, a fundamental difference is that in the tâtonnement process, there is a single public price whose variations induce massive side effects whereas in our setting price changes are individual and, taken individually, only have a marginal impact on the economy.

## 7 Extension to an Arbitrary Economy

Propositions 2 and 3 require that the goods an agent buys and sells be fixed independently of relative prices. This assumption can be relaxed by localizing the notions of acceptable buyers and sellers defined in the preceding section. In the following, we do not assume that assumption 3 holds and consequently adapt the definitions and proofs of the preceding section (incidentally, we slightly overload some of the notations). Given a population  $\pi \in P^m$ , we define:

- the set of buyers of good  $h$  as  $B_h(\pi) := \{i \mid d_{i,h}(\pi_i) > e_{i,h}\}$ ;
- the set of sellers of good  $h$  as  $S_h(\pi) := \{i \mid d_{i,h}(\pi_i) \leq e_{i,h}\}$ ;
- the set of acceptable buyers as those agents whose prices is above the lowest selling price that is  $\mathcal{B}_h(\pi) := \{i \in \{1, \dots, m\} \mid \pi_{i,h} \geq \min_{j \in S_h(\pi)/\{i\}} \pi_{j,h}\}$ ;

- the set of acceptable sellers as those agents whose prices is below the highest buying price, that is  $\mathcal{S}_h(\pi) := \{i \in \{1, \dots, m\} / \{i\} \mid \pi_{j,h} \leq \max_{i \in B_h(\pi)} \pi_{i,h}\}$ ;
- the feasible income as  $w_i(\pi) = \sum_{\{h \mid i \in \mathcal{S}_h(\pi)\}} \pi_{i,h} e_h$ .

The key differences with the preceding section is that here an acceptable buyer (resp. seller) is not necessarily a buyer (resp. seller). In particular, at an uniform price profile, each agent is both an acceptable buyer and an acceptable seller. In fact, whether an agent will end up being a net buyer or a net seller for a given good doesn't depend solely on his excess demand but on the complete demand profile in the economy. In other words, we assume that an agent can't restrict his trades to fulfill in priority his own excess demand. He must bring all his endowment to the market and let the exchange process determine the allocation. The set of price feasible allocations  $\mathcal{A}''(e, \pi)$  is then defined in a similar way as in definition (1), but for the fact that conditions bear on excess demands rather than on demands.

**Definition 2** *The set of price feasible allocations  $\mathcal{A}''(e, \pi)$  is the subset of feasible allocations  $\mathcal{A}(e)$  such that for every  $x \in \mathcal{A}''(e, \pi)$  one has:*

1.  $\forall h, \forall i \in B_h(\pi), x_{i,h} > e_{i,h} \Rightarrow i \in \mathcal{B}_h(\pi)$ ;
2.  $\sum_{i \in \mathcal{B}_h(\pi) \cap B_h(\pi)} (x_{i,h} - e_{i,h}) \leq \sum_{j \in \mathcal{S}_h(\pi)} (e_{j,h} - x_{j,h})$
3.  $x_i \leq d_i(\pi_i, w_i(\pi))$ ;

Assumption (4) also has an exact counterpart:

**Assumption 7 (Monotonicity Bis)** *If  $\pi$  and  $\pi'$  are such that  $\mathcal{A}''(e, \pi) \subset \mathcal{A}''(e, \pi')$  then for all  $i = 1 \dots n$ , one has  $u_i(\xi_i(\pi')) \geq u_i(\xi_i(\pi))$ .*

Assumptions 5, 6 and 7 then suffice to establish the counterpart of proposition 2 but for the two following caveats. First, the definition of acceptable sellers and buyers prevent an agent from being the sole buyer and seller of a given good. Therefore, we have to assume that (at least at equilibrium) there are at least a buyer and a distinct seller for every good. Second, in our framework the change of the private price of a commodity he neither consumes nor is endowed with has no effect whatsoever on an agent's utility, so that two strategies that differ only for such a good yield exactly the same utility. This might prevent the identification of Walrasian equilibria with strict equilibria. In order to avoid this failure, one shall assume that either each agent consumes or is endowed with each good or that an agent's strategy space is reduced to meaningful prices: these of commodities he consumes or sells. Assuming both conditions fulfilled, we can proceed with the proof of the following proposition, which is very similar to that of proposition 2 and hence given in Appendix B.

**Proposition 4** *Under Assumptions 5, 6 and 7,  $\bar{\pi} \in \Pi$  is a strict Nash equilibrium of  $\mathcal{G}(u, e, \xi)$  if and only if there exists  $\bar{p} \in \mathbf{E}(u, e)$  such that for all  $i \in \{1, \dots, m\}$ ,  $\bar{\pi}_i = \bar{p}$ .*

## 8 Characterization of Free Competition

It remains to analyze how restrictive is assumption 6 about buyers' incentive to increase prices when there is excess demand. Walras' description of buyers raising their bids is the report of an empirical observation, the "natural" expression of competition. We shall investigate here how this idea of competition can be grounded in the exchange process  $\xi$ .

Let us first consider a basic example with two agents and two goods. The first agent derives utility from consumption of good 2 only, e.g his utility function is  $u_1(x_1, x_2) = x_2$ , and is endowed with a quarter unit of good 1, i.e his initial endowment is  $e_1 = (1/4, 0)$ . The second agent has Cobb-Douglas preferences  $u_2(x_1, x_2) = x_1x_2$  and initial endowment  $e_2 = (1/4, 3/4)$ . Good 2 is the numeraire and its price is fixed equal to one. It is straightforward to check that the only equilibrium is such that the price equals  $(1, 1)$ , agent 1 is allocated  $(0, 1/4)$  and agent 2 is allocated  $(1/2, 1/2)$ . It is also clear that if both agents adopt  $(1, 1)$  as private price, the only efficient price feasible allocation is the equilibrium one. Yet, if both agents adopt  $(p, 1)$  as private price, then the demands of agent 1 and 2 respectively are  $d_1(p) = (0, p/4)$  and  $d_2(p) = (1/8 + 3/8p, p/8 + 3/8)$ . Whenever  $p < 1$ , there is excess demand for good 1 and the only efficient allocation is  $(0, p/4)$  to agent 1 and  $(1/2, 3-p/4)$  to agent 2. Let us examine assumption 6 in this setting. Agent 1 who is not rationed would be worse off if he decreased his private price for good 1 and hence its income. Also he cannot increase it unilaterally as he would price himself out of the market. Agent 2 cannot decrease his private price for good 1 as he would price himself out of the market. One would expect that "competition" induces him to increase his price for good 1 but he has no incentive to do so as this would only decrease his purchasing power and hence his utility. The key issue is that agent 2 actually faces no competition on the good 1 market as there is no other buyer he could outbid by increasing his price.

Let us then consider a more competitive situation by "splitting in two" agent 2. That is we consider an economy with three agents and two goods. The first agent still has utility function  $u_1(x_1, x_2) = x_2$ , and initial endowment  $e_1 = (1/4, 0)$ . The second and third agent have Cobb-Douglas preferences  $u_2(x_1, x_2) = u_3(x_1, x_2) = x_1x_2$  and initial endowment  $e_2 = e_3 = (1/8, 3/8)$ . As above, It is straightforward to check that the only equilibrium is such that the price equals  $(1, 1)$ , agent 1 is allocated  $(0, 1/4)$  while agents 2 and 3 are allocated  $(1/4, 1/4)$ . It is also clear that if

both agents adopt  $(1, 1)$  as private price, the only efficient price feasible allocation is the equilibrium one. Now, if each agent adopts  $(p, 1)$  as private price, then the demands respectively are  $d_1(p) = (0, p/4)$  and  $d_2(p) = d_3(p) = (1/16 + 3/16p, p/16 + 3/16)$ . Whenever  $p < 1$ , there is excess demand for good 1. It seems natural to assume that the exchange process would then allocate its demand  $d_1(p) = (0, p/4)$  to agent 1 who is not rationed. As far as agents 2 and 3 are concerned, the allocation is efficient provided they are allocated no more than their demand  $1/16 + 3/16p$  in good 1. It seems sensible to consider that the allocation is symmetric and hence that both agents are allocated  $(1/4, 3-p/8)$  and so are rationed in good 1. As before agent 1 has no incentive to change his private price for good 1 and neither agents 2 nor 3 can further decrease their private price for good 1. However, if the exchange process implements a form of competition between buyers by fulfilling in priority the demand of the agent offering the highest price for the good, then agents 2 and 3 have an incentive to increase their private prices for good 1. Indeed assume that agent 2 increases his private price for good 1, so  $q > p$ . He will then be allocated his demand  $(1/16 + 3/16q, q/16 + 3/16)$  and be better off than at price  $p$  provided that  $(1/16 + 3/16q)(q/16 + 3/16) > 1/4 \cdot 3-p/8$ . It is straightforward to check that this equation holds for  $q = p$  (whenever  $p < 1$ ) and hence by continuity in a neighborhood of  $p$ . In particular, there exists  $q > p$  such that agent 2 is better off adopting  $q$  as private price for good 1.

The above examples show that necessary conditions for assumption 6 to hold are potential and actual competition among buyers, which are respectively ensured *via* the presence of more than one buyer for every good and the priority given to highest bidding buyers in the trading process. It turns out these two conditions are in fact sufficient.

Since Edgeworth (1881), notably in (Debreu 1963), the seminal way to ensure competition is effective in a general equilibrium economy is to consider that the economy consists of sufficiently many replicates of a given set of primitive types. For our purposes, it suffices to assume that the economy  $\mathcal{E}(u, e)$  is a 2-fold replicate of some underlying simple economy.

**Assumption 8 (Replicates)** *For every  $i \in \{1 \cdots m\}$  there exists  $i' \in \{1 \cdots m\} \setminus \{i\}$  such that  $u_i = u_{i'}$  and  $e_i = e_{i'}$ . Types  $i$  and  $i'$  are called replicates.*

In our framework, a companion assumption is to consider that the exchange process is symmetric with respect to replicates using the same private price. That is:

**Assumption 9 (Symmetry)** *For every  $\pi \in P^m$ , if  $i, i' \in \{1 \cdots m\}$  are replicates such that  $\pi_i = \pi_{i'}$ , then  $\xi_i(\pi) = \xi_{i'}(\pi)$ .*

Then, to ground in the exchange process  $\xi$  actual effects of competition, we shall assume that if there is excess demand for one good the highest bidding seller has priority access to the market. That is a buyer who deviates upwards from a uniform price profile has its demand fulfilled in priority.

**Assumption 10 (High Bidders Priority)** *Let  $\pi = (p, \dots, p) \in P^m$  be a uniform price profile,  $i, i' \in \{1, \dots, m\}$  be replicates and  $\pi'$  a price profile such that  $\pi'_{i,h} > p_h$  for every  $h \in C_i$  and  $\pi_{j,k} = p_k$  otherwise. Then defining*

$$X_i(\pi') = \{x_i \in \mathbf{R}_+^l \mid x_i \leq \xi_i(\pi) + \xi_{i'}(\pi) \text{ and } \pi'_i x_i \leq \pi'_i e_i\},$$

we have

$$u_i(\xi_i(\pi')) \geq \max_{x_i \in X_i(\pi')} u_i(x_i).$$

That is, given that agent  $i$  gains priority over his replicate by increasing his price, everything goes as if he could pick any allocation satisfying his private budget constraint in the pool formed by adding the allocations he and his replicates were formerly allocated.

As announced, the latter conditions suffice to ensure that competition holds in the sense of assumption (6). Namely, one has

**Proposition 5** *If assumptions 3, 4, 8, 9, and 10 hold, then assumption 6 holds.*

**Proof:** *Let  $\pi = (p, \dots, p) \in P^m$  be a uniform price profile such that for some  $h \in \{1, \dots, l\}$ , one has  $\sum_{i=1}^m d_{i,h}(p, w_i(\pi)) > \sum_{i=1}^m e_{i,h}$ . We shall prove that there exists  $p' \in P$  such that  $u_i(\xi_i(p', \pi_{-i})) \geq u_i(\xi_i(\pi))$ . If there exists  $i$  such that for some  $h \in C_i$ ,  $\xi_{i,h}(\pi) = 0$ , the proof is straightforward according to assumption (3). Otherwise, let us then consider an agent  $i$  such that  $\xi_{i,h}(\pi) < d_{i,h}(\pi_i, w_i(\pi))$ . It is then clear, given condition 3 in the definition of  $\mathcal{A}'$ , that  $u_i(\xi_i(\pi)) < u_i(d_{i,h}(\pi_i, w_i(\pi)))$ . Hence it must either be that:*

- *the private budget constraint of agent  $i$  is not binding, that is one has  $p \cdot \xi_i(\pi) < p \cdot e_i$ ,*
- *or there are budget neutral utility improving shifts in consumption, that is there exists  $v \in \mathbf{R}_+^l$  such that  $v_h = 0$  for all  $h \notin C_i$  and  $p \cdot v = 0$  such that for all sufficiently small  $t > 0$ ,  $u_i(\xi_i(\pi) + tv) > u_i(\xi_i(\pi))$ .*

*In the case where agent  $i$  private budget constraint is not binding it is clear, given assumption 10, that agent  $i$  can still afford and obtain  $\xi_i(\pi)$  if he shifts to a price  $p' \in P$  such that  $p'_h > p_h$  for every  $h \in C_i$ ,  $p'_h \geq p_h$  otherwise and  $p'$  is*

sufficiently close to  $p$ . In the latter case where for all sufficiently small  $t > 0$ , one has  $u_i(\xi_i(\pi) + tv) > u_i(\xi_i(\pi))$ , denoting by  $i'$ , the replicate of  $i$  (who receives the same allocation according to assumption (9)) one has for  $t > 0$  sufficiently small:  $(\xi_i(\pi) + tv) \leq \xi_i(\pi) + \xi_{i'}(\pi)$ . Then for any  $p'$  such that  $p'_h > p_h$  for every  $h \in C_i$ , and  $p'_h \geq p_h$ , let us set  $x_i(p') = \frac{p' \cdot e_i}{p \cdot e_i}(\xi_i(\pi) + tv)$ . One clearly has  $x_i(p') \leq \xi_i(\pi) + \xi_{i'}(\pi)$  and  $p' \cdot x_i(p') \leq p' \cdot e_i$ . This implies according to assumption (10) that  $\xi_i(\pi') \geq u_i(x_i(p'))$  (where  $p'$  is defined as in assumption (10)). Moreover, for  $p'$  sufficiently close to  $p$ , one has using the continuity of the utility that  $u_i(x_i(p')) > u_i(\xi_i(\pi))$ . This ends the proof.

## 9 Conclusion

We have shown that the general equilibrium of a Walrasian market system is the only strict Nash equilibrium of an exchange game in which the requirements of the exchange process are quite mild and easily satisfied. Assuming producers update their private price profiles periodically by adopting the strategies of more successful peers, we have a multipopulation game in which strict Nash equilibria are asymptotically stable in the replicator dynamic. Conversely, all stable equilibria of the replicator dynamic are strict Nash equilibria of the exchange process and hence Walrasian equilibria of the underlying economy.

The major innovation of our model is the use of private prices, one set for each agent, in place of the standard assumption of a uniform public price faced by all agents, and the replacement of the tâtonnement process with a replicator dynamic. The traditional public price assumption would not have been useful even had a plausible stability theorem been available using such prices. This is because there is no mechanism for prices to change in a system of public prices—no agent can alter the price schedules faced by the large number of agents with whom any one agent has virtually no contact.

The private price assumption is the only plausible assumption for a fully decentralized market system not in equilibrium, because there is in fact no natural way to define a common price system except in equilibrium. With private prices, each individual is free to alter his price profile at will, market conditions alone ensuring that something approximating a uniform system of prices will prevail in the long run.

There are many general equilibrium models with private prices in the literature, based for the most part on strategic market games (Shapley and Shubik 1977, Sahi and Yao 1989, Giraud 2003) in which equilibrium prices are set on a market-by-market basis to equate supply and demand, and it is shown that under appropriate conditions the Nash equilibria of the model are Walrasian equilibria. These are

equilibrium models, however, without known dynamical properties, and unlike our approach they depend on an extra-market mechanism to balance demand and supply.

The equations of our dynamical system are too many and too complex to solve analytically or to estimate numerically. However, it is possible to construct a discrete version of the system as a finite Markov process. The link between stochastic Markov process models and deterministic replicator dynamics is well documented in the literature. For sufficiently large population size, the discrete Markov process captures the dynamics of the Walrasian economy extremely well with near certainty (Benaim and Weibull 2003). The dynamics of the Markov process model can be studied for various parameter values by computer simulation (Gintis 2007, 2012), whereas analytical solutions for the discrete system cannot be practically implemented, although they exist (Kemeny and Snell 1960, Gintis 2009).

Macroeconomic models have been especially handicapped by the lack of a general stability model for competitive exchange. The proof of stability of course does not shed light on the fragility of equilibrium in the sense of its susceptibility to exogenous shocks and its reactions to endogenous stochasticity. These issues can be studied directly through Markov process simulations, and may allow future macroeconomists to develop analytical microfoundations for the control of excessive market volatility.

## Appendix A: Asymptotic stability and replicator dynamics

Let  $G$  be an  $n$ -player game with finite strategy sets  $\{S_i | i = 1, \dots, n\}$ , the cardinal of which is denoted by  $k_i = |S_i|$ , with strategies indexed by  $h = 1, \dots, k_i$  and payoff functions  $\{\pi_i | i = 1, \dots, n\}$ . Let  $\Delta_i = \{\sigma_i \in \mathbf{R}^{k_i} | \forall h, \sigma_{i,h} \geq 0 \text{ and } \sum_{h=1}^{k_i} \sigma_{i,h} = 1\}$  the which is the mixed strategy space of agent  $i$ , and let  $\Delta = \prod_{i=1}^m \Delta_i$ . In an evolutionary game setting, an element  $\sigma_i \in \Delta_i$  represents a population of players  $i$  with a share  $\sigma_{i,h}$  of the population playing strategy  $h \in S_i$ .

Dynamics for such population of players  $(\sigma_1, \dots, \sigma_N) \in \Delta$  are defined by specifying, a growth rate function  $g : \Delta \rightarrow \mathbf{R}^{\sum_{i=1}^m k_i}$ , for all  $i = 1, \dots, n$  and  $h = 1, \dots, k_i$ :

$$\frac{\partial \sigma_{i,h}}{\partial t} = \sigma_{i,h} g_{i,h}(\sigma) \quad (2)$$

We shall restrict attention to growth-rate functions that satisfy a regularity condition and maps  $\Delta$  into itself (Weibull 1995).

**Definition 3** *A regular growth-rate function is a Lipschitz continuous function  $g$  defined in a neighborhood of  $\Delta$  such that for all  $\sigma \in \Delta$  and all  $i = 1, \dots, n$  we have  $g_i(\sigma) \cdot \sigma_i \neq 0$ .*



The dynamics of interest in a game-theoretic setting are those that satisfy minimal properties of monotonicity with respect to payoffs. Strategies of player  $i$  in  $B_i(\sigma) := \{s \in S_i | u_i(s, \sigma_{-i}) > u_i(\sigma)\}$  that have above average payoffs against  $\sigma_{-i}$ , have a positive growth-rate in the following sense:

**Definition 4** A regular growth-rate function  $g$  is weakly payoff-positive if for all  $\sigma \in \Delta$  and  $i = 1, \dots, n$ ,

$$B_i(\sigma) \neq \emptyset \Rightarrow g_{i,h} > 0 \text{ for some } s_{i,h} \in B_i(\sigma), \quad (3)$$

where  $s_{i,h}$  denotes the  $h^{\text{th}}$  pure strategy of player  $i$ .

Among the class of weakly-payoff positive dynamics, the replicator dynamic is by far the most commonly used to represent the interplay between population dynamics and strategic interactions. It corresponds to the system of differential equations defined for all  $i = 1, \dots, n$  and  $h = 1, \dots, |S_i|$  by:

$$\frac{\partial \sigma_{i,h}}{\partial t} = \sigma_{i,h} (\pi_i(s_{i,h}, \sigma_{-i}) - \pi_i(\sigma)). \quad (4)$$

That is thus the system of differential equation corresponding to the growth rate function  $g_{i,h}(\sigma) = \pi_i(s_{i,h}, \sigma_{-i}) - \pi_i(\sigma)$ .

It is standard to show that the system of differential equations (2) associated with a regular and weakly-payoff monotonic growth function has a unique solution defined at all times for every initial condition in  $\Delta$ . We will generically denote the solution mapping by  $\psi : \mathbf{R}_+ \times \Delta \rightarrow \Delta$ , so  $\psi(t, \sigma_0)$  gives the value at time  $t$  of the solution to (2) with initial condition  $\sigma(0) = \sigma_0$ . Stability properties of (2), are then defined in terms of this solution mapping:

**Definition 5** A strategy profile  $\sigma^* \in \Delta$  is called Lyapunov stable if every neighborhood  $V$  of  $\sigma^*$  contains a neighborhood  $W$  of  $\sigma^*$  such that  $\psi(t, \sigma) \in V$  for all  $\sigma \in W \cap \Delta$ .

**Definition 6** A strategy profile  $\sigma^* \in \Delta$  is called asymptotically stable if it is Lyapunov stable and there exists a neighborhood  $V$  of  $\sigma^*$  such that for all  $\sigma \in V \cap \Delta$ :

$$\lim_{t \rightarrow +\infty} \psi(t, \sigma) = \sigma^*.$$

## Appendix B: Proof of proposition 4.

The proof of proposition 4 proceeds as follows.

**Proof:** According to assumption 5, if  $\bar{\pi} = (\bar{p}, \dots, \bar{p})$  is an equilibrium price profile then for all  $i$ ,  $\xi_i(\bar{\pi}) = d_i(\bar{p}, w_i(\bar{\pi}))$ , where  $w_i(\bar{\pi}) = \bar{p} \cdot e_i$ . Assume then agent  $i$  deviates to a price  $p \neq \bar{p}$  and let  $\pi' = (p, \pi_{-i})$ . Then, one has:

- If there is  $h$  such that  $p_h > \bar{p}_h$ , one has  $i \notin \mathcal{S}_h(\pi')$  and agent  $i$  can no longer sale good  $h$  and does not rise any income on the good  $h$  market.
- If there is  $h$  such that  $p_h < \bar{p}_h$ , one clearly has  $p_h e_{i,h} < \bar{p}_h e_{i,h}$ .

Given that at the equilibrium price profile  $\bar{\pi}$ , every agent is an acceptable seller for every good so that  $w_i(\bar{\pi}) = \sum_{h=1}^{\ell} \pi_{i,h} e_{i,h}$ , it follows that  $w_i(\pi') \leq w_i(\bar{\pi})$ , the inequality being strict unless  $e_{i,h} = 0$  for every  $h$  such that  $p_h \neq \bar{p}_h$ .

If the latter condition holds, there must be  $h$  such that  $p_h \neq \bar{p}_h$  and  $e_{i,h} = 0$  (so that for any price profile  $\rho$ ,  $i \in B_h(\rho)$ ). For any  $h$  such that  $p_h < \bar{p}_h$ , then  $i \notin \mathcal{B}_h(\pi')$  and one necessarily has  $\xi_{i,h}(p', \pi_{-i}) = 0$  according to the definition of  $\mathcal{A}$  (and given that  $e_{i,h} = 0$ ). Hence agent  $i$  can only consume goods  $h$  such that  $p_h \geq \bar{p}_h$ .

To sum up, if agent  $i$  deviates to price  $p'$  his income and his choice set are both reduced and one of them is strictly reduced. He cannot by definition obtain an allocation as good as  $d_i(\bar{p}, \bar{p} \cdot e_i)$  and hence is strictly worse off.

Suppose then  $\pi$  is not an equilibrium price profile.

If  $\pi$  is a uniform price profile and there exists an  $h \in \{1, \dots, l\}$  such that  $\sum_{i=1}^m d_{i,h}(p, w_i(\pi)) > \sum_{i=1}^m e_{i,h}$ , then assumption 6 implies  $\pi$  is not a (strict) Nash equilibrium.

Hence, from here on we can assume that  $\pi$  is not a uniform price profile. Then:

- If there exists  $h \in \{1, \dots, \ell\}$  and  $i \in \mathcal{S}_h(\pi)/\mathcal{S}_h(\pi)$ , by setting  $p'_h := \max_{i \in B_h(\pi)/\{i\}} \pi_{i,h}$  and  $p'_j := \pi_{i,j}$  for all  $j \neq h$  one obtains a population  $\pi' := (p', \pi_{-i})$  such that  $w_i(\pi') \geq w_i(\pi)$  while the other constraints in the definition of  $\mathcal{A}''$  can only be relaxed. Therefore  $\mathcal{A}''(e, \pi) \subset \mathcal{A}''(e, \pi')$ . According to assumption 7, it can not be that  $\pi$  is a strict Nash equilibrium. Similar arguments apply whenever there exists  $h \in \{1, \dots, \ell\}$  and  $i \in B_h(\pi)/\mathcal{B}_h(\pi)$ .
- Otherwise, one must have  $B_h(\pi) \subset \mathcal{B}_h(\pi)$ ,  $\mathcal{S}_h(\pi) \subset \mathcal{S}_h(\pi)$ , and there must exist  $i, i' \in \{1, \dots, m\}$  and  $h \in \{1, \dots, l\}$  such that  $\pi_{i,h} \neq \pi_{i',h}$ , e.g.  $\pi_{i,h} < \pi_{i',h}$ . One then has:

- If  $i, i' \in S_h(\pi)$  by setting  $\pi'_{i,h} = \pi_{i',h}$  and  $\pi'_{j,k} = \pi_{j,k}$  for  $j \neq i'$  or  $k \neq h$ , one has  $w_i(\pi') \geq w_i(\pi)$  while the other constraints in the definition of  $\mathcal{A}''$  can only be relaxed. Therefore  $\mathcal{A}''(e, \pi) \subset \mathcal{A}''(e, \pi')$  so that  $\pi$  can not be a strict Nash equilibrium according to assumption 7. The same holds true if  $i \in S_h(\pi)$  and  $i' \in B_h(\pi)$ .

Hence, from here on we can assume that for all  $i, i' \in S_h$ ,  $\pi'_{i,h} = \pi_{i',h}$ . Then:

- If  $i, i' \in B_h(\pi)$  by setting  $\pi'_{i',h} = \pi_{i,h}$  and  $\pi'_{j,k} = \pi_{j,k}$  for  $j \neq i$  or  $k \neq h$ , the private budget constraint of agent  $i'$  (condition 3 in the definition of  $\mathcal{A}''$ ) is relaxed and one then has  $\mathcal{A}''(e, \pi) \subset \mathcal{A}''(e, \pi')$  so that  $\pi$  can not be a strict Nash equilibrium according to assumption 7.
- If  $i \in B_h(\pi)$  and  $i' \in S_h(\pi)$ , it must be, given that  $S_h(\pi) = \mathcal{S}_h(\pi)$ , that there exist  $i'' \in B_h$  such that  $\pi_{i',h} \leq \pi_{i'',h}$  and the preceding argument applies to  $i, i'' \in B_h(\pi)$  such that  $\pi_{i,h} < \pi_{i'',h}$ .

Summing up,  $\pi$  cannot be a strict Nash equilibrium of  $\mathcal{G}(u, e, \xi)$  if it is not an equilibrium price profile in the Walrasian sense. This ends the proof.

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