Stochastic stability in the Scarf economy

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\textbf{HIGHLIGHTS}

- Analyzes the best-known example of tâtonnement instability: the Scarf Economy.
- Introduces a novel approach using bargaining games and stochastic stability.
- Shows stochastic stability of equilibrium in the Scarf Economy.
- Bridges simulation and analytical models.
- Paves the way for a general theory of evolutionary stability of equilibrium.

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\textbf{ABSTRACT}

We present a mathematical model for the analysis of the bargaining games based on private prices used by Gintis to simulate the dynamics of prices in exchange economies in Gintis (2007). We then characterize, in the Scarf economy, a class of dynamics for which the Walrasian equilibrium is the only stochastically stable state. Hence, we provide dynamic foundations for general equilibrium for one of the best-known examples of instability of the tâtonnement process.

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1. Introduction

The Scarf economy Scarf (1960) is the paradigmatic example of the failure of the Walrasian tâtonnement process to provide a generically valid model of the convergence of an economy to its general equilibrium. Accordingly, it has served as the test-case against which to assess alternative models of price adjustment (see e.g. Kumar and Shubik, 2004; Crockett, 2013; Ghosal and Porter, 2013, in the recent literature). More broadly, the literature has tried to address the issues raised by Scarf’s example through two main lines of research. The first, in line with Scarf’s own concern has focused on the issue of calculability of equilibrium (see e.g. Scarf, 1969; Herings, 1997) and more recently, somehow as a by-product of the surge of results in algorithmic game theory, on its approachability (see Papadimitriou and Yannakakis, 2010). The second has taken a more behavioral route and has seek to develop micro-foundations for out-of-equilibrium dynamics. In this respect Fisher (1983) gives an account of the initial results obtained via non-tâtonnement mechanisms. Giraud (2003) surveys approaches based on market-games, Serrano and Volij (2008) or Vega-Redondo (1997) provide evolutionary approaches, and Crockett (2013) gives a survey of experimental results.

It seems that the two strands of literature have never converged again. That is the non-tâtonnement literature has abandoned the objective of providing dynamic micro-foundations for the Walrasian equilibrium of an economy in favor of a much more modest one, which is to show convergence to a stable and generally Pareto optimal outcome. This program remains at odds with the central role assumed by the concept of general equilibrium in most of the theoretical and applied literature.

As a potential contribution to the resolution of this dilemma, the aim of this note is to exhibit, in the framework of the Scarf economy, an approach of equilibrium dynamics that is both micro-founded and well-behaved asymptotically. Given its remarkable status, we assume that the Scarf economy is a good starting point to motivate further generalizations. Our model is based on a series of recent contributions (see Gintis, 2007, 2012), which have revisited the issue of “walrasian dynamics” using computer simulations where agents repeatedly perform the following sequence of operations: they receive their initial endowment, engage in bilateral trades on the basis of private prices, consume, and update their private prices on the basis of the utility these prices yielded during the period. Put differently, we focus on evolutionary dynamics in bargaining games played by agents who use private prices as strategies. This approach harnesses the power of evolutionary dynamics both as a model of economic behavior and as a computational paradigm. As a matter of fact, Gintis (2007,
randomly mutate them in some instances. Hasapri... ensure the stochastic stability of trading in the Scarfe economy and gives sufficient conditionson of Ellison (2000). In Section 3, we characterize out-of-equilibrium Markov chain structure of Gintis’ evolutionary bargaining models under-explored field of study.

The paper is organized as follows: in Section 2, we explicit the behavior observed in Gintis’s simulations. The formalism we develop might also pave the way for the proof of more general results of convergence to general equilibrium in evolutionary models, for which related contributions (see Serrano and Volij, 2008; Vega-Redondo, 1997) provide evidence. An important caveat however is that although our result focuses on dynamic aspects of price formation, they are valid only in a setting with steady and non-durable resources, that is in the absence of opportunities to transfer wealth across periods or to make intertemporal choices. Out-of-equilibrium dynamics in growth models certainly is an under-explored field of study.

The paper is organized as follows: in Section 2, we explicit the Markov chain structure of Gintis evolutionary bargaining models and show they are models of evolution with noise in the sense of Ellison (2000). In Section 3, we characterize out-of-equilibrium trading in the Scarf economy and give sufficient conditions on the price updating mechanism to ensure the stochastic stability of equilibrium. Section 4 offers our conclusion.

2. Evolutionary dynamics in exchange economies

We aim at investigating evolutionary dynamics in exchange economies where each agent carries a private vector of prices (i.e has a private valuation of goods), uses these private prices in order to determine acceptable trades, update them by imitating those of peers who were more successful in the trading process, and randomly mutate them in some instances.

More precisely, let us consider an exchange economy with L goods,1 N types of agents2 and M agents of each type.3 All the agents have $Q := \mathbb{R}^L$ as consumption set. Agents of type i are characterized by a utility function $u_i : Q \to \mathbb{R}$ and a vector of initial endowment $\omega_i \in Q$. Moreover, agent $(i, j)$ (the jth agent of type i) is endowed with a normalized vector $p_{i,j}$ of private prices chosen in a finite subset $P$ of the unit simplex of $\mathbb{R}^L_+$, $S := \{ p \in \mathbb{R}^L_+ \mid \sum_{l=1}^L p_l = 1 \}$. The population of agents is then characterized by a vector $\pi \in \Pi = \mathbb{R}^{NM \times N}$.

Repeated bilateral trades between agents define a trading process, which allocates as a function of agents private prices the total resources of the economy. This process might involve some randomness in order to cope with rationing in out-of-equilibrium situations. In all generality, we can represent the trading process by a transition measure $\mathcal{T}$ from $\Pi$ to $\mathcal{Z}$ which associates to a population of prices $\pi \in \Pi$, a probability distribution $T_\pi$ on the set of allocations $\mathcal{Z}$ defined as

$$\mathcal{Z} = \left\{ \xi \in \mathbb{Q}^{NM} \mid \sum_{i=1}^N \sum_{j=1}^M \xi_{i,j} = N \sum_{i=1}^N \omega_i \right\}. \quad (1)$$

where $\xi_{i,j}$ represents the allocation to the jth agent of type i.

Private prices are then updated through an imitation process: agents imitate peers of the same type taking into consideration the utility gained through trading. In all generality, we can represent this imitation process as associating to a population of prices $\pi \in \Pi$ and to an allocation $\xi \in \mathcal{Z}$, a probability distribution $T_\pi(\xi)$ on $\mathcal{Z}$ (which gives the distribution of prices after updating).

We are then concerned with the dynamics of private prices generated by the sequential iteration of trading and imitation processes. That is the process in which: initial endowments are reinitialized at the beginning of each step, agents trade according to their private prices and update these as a function of the utility gained. This corresponds to the Markovian dynamics on $\mathcal{Z}$ defined by the transition matrix $F$ such that

$$F_{\pi,\pi'} = \int_{\xi \in \mathcal{Z}} T_\pi(\xi) \, d\mathcal{T}_\pi(\xi). \quad (2)$$

If agents then randomly and independently mutate (i.e randomly choose a new price in $P$) with probability $\epsilon > 0$, the dynamics are modified according to

$$F_\epsilon^\pi = \int_{\rho \in \Pi} R_{\rho,\pi} \, dF_{\rho,\pi} = \sum_{\rho \in \Pi} R_{\rho,\pi} F_{\rho,\pi} \quad (3)$$

where $R_{\rho,\pi} = (1 - e^{MN - \delta(\rho, \pi)}) \cdot \left( \frac{e \cdot 1}{\pi \cdot 1} \right)^{\delta(\rho, \pi)}$ and $\delta(\rho, \pi)$ denotes the number of mutations, that is the cardinal of the set \{(i, j) \mid p_{i,j} \neq p'_{i,j}\}.

The family $(F_\epsilon^\pi)_{\epsilon \geq 0}$ then is a model of evolution in the sense of Ellison (2000), that is satisfies the following conditions:

1. $F_\epsilon^\pi$ is ergodic for each $\epsilon > 0$.
2. $F_\epsilon^\pi$ is continuous in $\epsilon$ and $F_0^\pi = F_\pi$.
3. there exists a function $c : \mathbb{P}^{NM} \times \mathbb{P}^{NM} \to \mathbb{N}$ such that for all $\pi, \pi' \in \mathbb{P}^{NM}$, $\lim_{\epsilon \to 0} F_\epsilon^\pi(\pi, \pi') \to c(\pi, \pi')$ exists and is strictly positive.

Condition (1) implies in particular that for each $\epsilon > 0$, $F_\epsilon^\pi$ has a unique invariant distribution $\psi_\epsilon$. A population $\pi \in \mathbb{P}^{NM}$ is then called stochastically stable if $\lim_{\epsilon \to 0} \psi_\epsilon(\pi) > 0$.

This notion of stochastic stability can be used for the analysis of the stability of the equilibria of the underlying exchange economy thanks to the identification of an equilibrium price $\overline{p}$ with the population $\Sigma$ such that every agent uses price $\overline{p}$ (that is such that for all $(i, j)$, one has $\pi_{i,j} = \overline{p}$). The equilibrium associated with the price $\overline{p}$ can then be called stochastically stable if $\pi$ is. The interesting case is this where $\pi$ is the only stochastically stable population which implies that $\lim_{\epsilon \to 0} \psi_\epsilon(\pi) = 1$ and that for vanishingly small perturbations the process eventually settles in $\pi$ independently of the initial conditions, in other words converges to equilibrium.

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1 Indexed by $i = 1 \cdots L$.
2 Indexed by $j = 1 \cdots N$.
3 Indexed by $k = 1 \cdots M$.
4 We shall assume that $\mathcal{Z}$ is endowed with the Borel $\sigma$-algebra.
5 This last point follows form the fact that the coefficients of $F_\epsilon(\pi, \pi')$ are polynomials in $\epsilon$. 

2012) report surprising results of convergence to equilibrium. We provide an analytical counterpart to these results.

Namely, we use the notion of stochastic stability (see Peyton-Young, 1993) to characterize the asymptotic properties of a stylized version of Gintis’s model. We show that the general equilibrium is the only stochastically stable state of this model. This implies that independently of the initial conditions, all the agents should eventually adopt the equilibrium price and obtain equilibrium allocations.

Our result mainly builds on the assumption that out-of-equilibrium trading is efficient in the same sense as in the Hahn process (see Hahn and Negishi, 1962): after trade there are not both unsatisfied suppliers and unsatisfied demanders for any given good. We also assume that agents strategically restrict their out-of-equilibrium trade whenever it is profitable to do so. In this setting, it turns out that price movement towards equilibrium are always favorable to a majority of agents. As “price-setting power” is uniformly distributed, given that each agent has its own private price to update, this progressively leads to the general adoption of the equilibrium price.

Hence, the main contribution of the paper is to explain the behavior observed in Gintis’s simulations. The formalism we develop might also pave the way for the proof of more general results of convergence to general equilibrium in evolutionary models, for which related contributions (see Serrano and Volij, 2008; Vega-Redondo, 1997) provide evidence. An important caveat however is that although our result focuses on dynamic aspects of price formation, they are valid only in a setting with steady and non-durable resources, that is in the absence of opportunities to transfer wealth across periods or to make intertemporal choices. Out-of-equilibrium dynamics in growth models certainly is an under-explored field of study.

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Repeated bilateral trades between agents define a trading process, which allocates as a function of agents private prices the
3. Stochastic stability in the Scarf economy

3.1. The economy

We shall investigate the stochastic stability of equilibrium in the Scarf (1960) economy, which is probably the best known example of non-stability of the tâtonnement process. In this economy, there are three goods and three types of agents whose respective utility, endowment and demand are as follows:

\[ u_1(x_1, x_2, x_3) = \min(x_1, x_2), \quad \omega_1 = (1, 0, 0), \]
\[ d_1(p_1, p_2, p_3) = \frac{p_1}{p_1 + p_2} (1, 1, 0) \]
\[ u_2(x_1, x_2, x_3) = \min(x_2, x_3), \quad \omega_2 = (0, 1, 0), \]
\[ d_2(p_1, p_2, p_3) = \frac{p_2}{p_2 + p_3} (0, 1, 1) \]
\[ u_3(x_1, x_2, x_3) = \min(x_1, x_3), \quad \omega_3 = (0, 0, 1), \]
\[ d_3(p_1, p_2, p_3) = \frac{p_3}{p_1 + p_3} (1, 0, 1). \]

This economy has a unique equilibrium for the price \( \bar{p} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) \). Whereas it is well-known that the law of demand does not hold and that the tâtonnement process does not converge but follows periodic orbits (see Negishi, 1962), simulations of evolutionary dynamics in the Scarf economy by Gintis (2007) suggest that convergence to equilibrium might hold for dynamics of the kind introduced in the preceding section. Accordingly, we provide below a proof of the stochastic stability of equilibrium in a stylized version of Gintis (2007) model.

3.2. Out-of-equilibrium trading

We first characterize the outcome of out-of-equilibrium trading in an economy with a single agent of each type and a public price. This characterization builds on two premises. First, we shall assume that trading is efficient in the sense put forward by the Hahn process (see Hahn and Negishi (1962) and Fisher (1983) for an in-depth discussion): after trade there are not both unsatisfied suppliers and unsatisfied demanders for any good given. Second, as in Gintis (2007), we consider that an agent allocates its initial endowment in priority to the fulfillment of its own demand at the expense of unfilled "outside" demand. Hence agents strategically restrict their out-of-equilibrium trade in order to maximize their own utility or, put differently, agents are not as myopic as to supply to a third party goods up to the point of being themselves rationed.

In the Scarf economy, the aggregate excess demand at a price \((p_1, p_2, p_3)\) is given by

\[ Z(p_1, p_2, p_3) = \left( \frac{p_1}{p_1 + p_2} + \frac{p_3}{p_1 + p_3} - 1, \frac{p_1}{p_1 + p_2} \right) \]
\[ + \frac{p_2}{p_2 + p_3} - 1, \frac{p_2}{p_2 + p_3} + \frac{p_3}{p_1 + p_3} - 1 \right) \]

and one has

- Excess demand for good 1 if \( \frac{p_1}{p_1 + p_2} + \frac{p_3}{p_1 + p_3} > 1 \) or equivalently \( p_3 > p_2 \);
- Excess demand for good 2 if \( \frac{p_1}{p_1 + p_2} + \frac{p_2}{p_2 + p_3} > 1 \) or equivalently \( p_1 > p_3 \);
- Excess demand for good 3 if \( \frac{p_2}{p_1 + p_2} + \frac{p_3}{p_1 + p_3} > 1 \) or equivalently \( p_2 > p_1 \).

The outcome of trading at price \((p_1, p_2, p_3)\) is then completely determined by the assumptions that trading is efficient and that agents strategically restrict their out-of-equilibrium trade. There are essentially three types of situation:

1. The price is such that there is rationing on two markets: for example, \( p \) satisfies \( p_2 > p_1 > p_3 \), so that there is rationing on good 2 and 3 markets. In this setting, agent 3 will not be rationed because he can strategically restrict his sales in order to fulfill its own demand of good 3 and because good 1 is not rationed at all. Hence, the trading process shall yield agent 3 the allocation \( \frac{p_1}{p_1 + p_3} (1, 0, 1) \). Agent 2 will then be rationed in good 3, as he might only be allocated the remaining \( \frac{p_1}{p_1 + p_3} \) units of good 3, rather than the \( \frac{p_2}{p_2 + p_3} \) units he demands. Consequently, agent 2 has no interest in retaining more than \( \frac{p_1}{p_1 + p_3} \) units of good 2 and can supply \( \frac{p_1}{p_1 + p_3} \) units of that good to agent 1. There are then two cases:
   (a) If \( \frac{p_1}{p_1 + p_3} \leq \frac{p_1}{p_1 + p_3} \), that is \( p_1^* \leq p_1^* p_2 \), agent 1 is rationed and the agents’ allocations and utilities are given by:

<table>
<thead>
<tr>
<th>Agent</th>
<th>Allocation</th>
<th>Utility</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \frac{p_1}{p_1 + p_3} (1, 0, 0) )</td>
<td>( \frac{p_1}{p_1 + p_2} )</td>
</tr>
<tr>
<td>2</td>
<td>( \frac{p_2}{p_2 + p_3} (1, 0, 1) )</td>
<td>( \frac{p_2}{p_2 + p_3} )</td>
</tr>
<tr>
<td>3</td>
<td>( \frac{p_3}{p_2 + p_3} (1, 0, 1) )</td>
<td>( \frac{p_3}{p_2 + p_3} )</td>
</tr>
</tbody>
</table>

2. The price is such that there is rationing on a single market, for example \( p \) satisfies \( p_1 > p_2 > p_3 \), so that there is rationing on good 2 market only. In this setting, agent 3 will not be rationed because there is no rationing in either of the good he demands, agent 2 will not be rationed either because there is no rationing in good 1 and he can strategically restrict its sales of good 2 in order to fulfill its own demand. The agents’ allocations and utilities are given by:

<table>
<thead>
<tr>
<th>Agent</th>
<th>Allocation</th>
<th>Utility</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( (1 - \frac{p_2}{p_1 + p_2}) \frac{p_1}{p_1 + p_3}, \frac{p_3}{p_1 + p_2}, 0 )</td>
<td>( \frac{p_1}{p_1 + p_2} )</td>
</tr>
<tr>
<td>2</td>
<td>( \frac{p_2}{p_2 + p_3} (0, 1, 1) )</td>
<td>( \frac{p_2}{p_2 + p_3} )</td>
</tr>
<tr>
<td>3</td>
<td>( \frac{p_3}{p_2 + p_3} (1, 0, 1) )</td>
<td>( \frac{p_3}{p_2 + p_3} )</td>
</tr>
</tbody>
</table>

Up to a permutation of indices, the other cases are similar. There is either rationing on two markets (if \( p_1 > p_2 > p_3 \) or \( p_1 > p_2 > p_3 \) or rationing on a single market (if \( p_1 > p_2 > p_3 \) or \( p_2 > p_3 \) or \( p_1 > p_2 > p_3 \)). One can then define the out-of-equilibrium allocation rule in the Scarf economy as the mapping \( \tilde{x}_1 : P \to Q^3 \) such that

\[ \tilde{x}_1(p_1, p_2, p_3) = \begin{cases} \left( \frac{p_1}{p_1 + p_3}, \frac{p_3}{p_1 + p_3}, 0 \right) & \text{if } p_2 > p_1 > p_3 \text{ and } p_2 > p_1 p_3 \\ \left( \frac{p_1}{p_1 + p_3}, \frac{p_3}{p_1 + p_3}, \frac{p_1}{p_1 + p_3} \right) - (p_2) & \text{if } p_1 > p_2 > p_3 \\ \left( \frac{p_1}{p_1 + p_3}, \frac{p_3}{p_1 + p_3}, \frac{p_1}{p_1 + p_3} \right) - (p_2) & \text{if } p_1 > p_2 > p_3 \\ \left( \frac{p_1}{p_1 + p_3}, \frac{p_3}{p_1 + p_3}, \frac{p_1}{p_1 + p_3} \right) - (p_2) & \text{if } p_1 > p_2 > p_3 \\ \left( \frac{p_1}{p_1 + p_3}, \frac{p_3}{p_1 + p_3}, \frac{p_1}{p_1 + p_3} \right) - (p_2) & \text{if } p_1 > p_2 > p_3 \\ \left( \frac{p_1}{p_1 + p_3}, \frac{p_3}{p_1 + p_3}, \frac{p_1}{p_1 + p_3} \right) - (p_2) & \text{if } p_1 > p_2 > p_3 \end{cases} \]
\[\bar{X}_{2,\sigma}(p_1, p_2, p_3) = \bar{X}_{1,\sigma}(p_0(1), p_0(2), p_0(3)) \]
\[\text{where } \sigma(1) = 2, \sigma(2) = 3, \sigma(3) = 1, \]
\[\bar{X}_{3,\tau}(p_1, p_2, p_3) = \bar{X}_{1,\tau}(p_1(1), p_2(1), p_3(1)) \]
\[\text{where } \tau(1) = 3, \tau(2) = 1, \tau(3) = 2. \]  

3.3. Trading process for a population of agents

We now want to define the outcome of out-of-equilibrium trading for a population of agents using private prices. In Gintis’s model, agents assess the value of trades using their private prices and perform those which have a non-negative value. However, tracking all such possible trades lead to excessive combinatorial complexity. In order to provide a parsimonious analysis, we rather consider the benchmark situation where agents only trade with peers using the same price. This restriction in fact discards “lucky” trades which would increase the value of one’s stock. It is also standard in the non-tâtonnement literature (see e.g. Fisher, 1983).

An alternative interpretation of this restriction of trading to peers using the same price is to consider that different market places or trading posts coexist, in each of which exchanges are performed according to one price, and that the private price of an agent is a marker of the market he is “affiliated” to. The choice of a private price by an agent can then be seen as a form of voting with one’s feet for a set of exchange ratios.

Formally, we place ourselves in the framework of Section 2, with \(N = 3\), utilities and endowments being those of the Scarf economy and we assume that \(P\) contains the equilibrium price \(\bar{p}\) = \((\frac{1}{3}, \frac{1}{3}, \frac{1}{3})\). We also let \(\mu_i(\pi, p)\) denote the number of agents of type \(i\) using price \(p\) in the population \(\pi\), \(\mu(\pi, p) = \sum_{i=1}^{N} \mu_i(\pi, p)\) denotes the total number of agents using price \(p\) in the population \(\pi\), and \(\mu(\pi, p) := \min(\mu_1(\pi, p), \mu_2(\pi, p), \mu_3(\pi, p))\) denotes the minimum number of agents using price \(p\) among the different types in the population \(\pi\).

In this setting the assumptions that agents only trade with peers using the same price and that they strategically restrict their trades amount to consider that agents of type \(i\) using price \(p\) collectively offer \(\mu_i(\pi, p)(c_0 - \bar{X}_i(p))\) units of good \(i\) to their peers of other types using price \(p\). To reduce the combinatorial complexity of the analysis, we shall restrict attention to the simplest allocations consistent with these constraints, namely those where \(\mu(\pi, p)\) among the agents using price \(p\) receive the allocation \(R(p)\), while the remaining retain their initial allocation and do not trade. The random component of the trading process is hence restricted to the designation of trading and non-trading agents. This amounts to assume that for all population \(\pi \in \Pi\) and type of agent \(i\), one has

\[\mathcal{F}_\pi \left\{ \begin{array}{l}
\xi \in \mathcal{S} \mid \text{card}\{j \mid \pi_{ij} = p \land \xi_{ij} = \bar{X}_i(p)\} = \mu(\pi, p) \\
\text{card}\{j \mid \pi_{ij} = p \land \xi_{ij} = \bar{X}_i(p)\} + 1 = 1. \end{array} \right. \]  

We hence obtain a simplified representation of the trading process of Gintis (2007), by assuming that trading is efficient and takes place only among subgroups consisting of a similar number of agents of each type.

3.4. Imitation process

In order to characterize the imitation process, we have to specify the probability of the event that the \(j\)th agent of type \(i\) adopts the price of the \(j\)th agent of type \(\bar{t}\). We denote this event by \(\{i, j \rightarrow (\bar{t}, j)\}\) and assume that for all pair of agents \((i, j)\) and \((\bar{t}, j)\), one has

\[\mathcal{I}(\pi, \xi) \{i, j \rightarrow (\bar{t}, j)\} \geq 0 \]

\[\mathcal{I}(\pi, \xi) \{i, j \rightarrow (\bar{t}, j)\} \geq 0 \]

Interpreted in conjunction with assumption (I), condition (II)(i) states that agents which have traded have a positive probability to copy the prices of a peer who has been more successful in trading.\footnote{We could as well assume that all agents can copy a peer’s price but we should then assume that they compare the utility each price yielded to agents which have actually traded.}

Condition (II)(ii) states that agents always have a positive probability to copy the most widely used price in the population. This prevents in particular that the system gets stuck in the extremely inefficient situation where each type of agent uses a different price.

Condition (II)(iii) states that an agent can conserve his price with some probability.

As far as the relationships between agents’ imitation behavior is concerned, we shall assume that there is independent inertia, that is the imitation probabilities are independent. Namely, for all agents \((i, j), (\bar{t}, j), (\bar{t}', j'), (\bar{t}'', j'')\) with \((i, j) \neq (\bar{t}, j')\), one has

\[\mathcal{I}(\pi, \xi) \{i, j \rightarrow (\bar{t}', j') \land (\bar{t}', j'') \rightarrow (\bar{t}'', j'')\} = \mathcal{I}(\pi, \xi) \{i, j \rightarrow (\bar{t}', j')\} \times \mathcal{I}(\pi, \xi) \{(\bar{t}', j'') \rightarrow (\bar{t}'', j'')\} . \]

3.5. Stochastic stability of equilibrium in the Scarf economy

Note that the population \(\pi\) where each agent uses the equilibrium price \(\bar{p}\) (i.e. for all \((i, j), \pi_{ij} = \bar{p}\) ) is an invariant distribution of the process \(\mathcal{F}\) under assumption (II) and that under assumption (I) it can naturally be identified with the equilibrium of the Scarf economy as the trading process \(\mathcal{F}\) then allocates to each agent the corresponding equilibrium allocation with probability one. Conditions (I), (II) and (III) are in fact sufficient to prove our main result.

Theorem 1. Under conditions (I), (II) and (III), the population \(\pi\) is the only stochastically stable state of the dynamics defined by \((\mathcal{F}, \varepsilon)_{\varepsilon \geq 0}\) in the Scarf economy.

Proof. The proof mainly builds on Ellison (2000) radius–coradius theorem, which makes use of the following notions:

- A path from \(\pi \in \mathbb{P}^{n \times m}\) to \(\pi' \in \mathbb{P}^{n \times m}\) is a finite sequence of states, \(\pi_1, \ldots, \pi_K \in \mathbb{P}^{n \times m}\), such that \(\pi_1 = \pi\) and \(\pi_K = \pi'\). The set of paths from \(\pi\) to \(\pi'\) is denoted by \(\mathcal{S}(\pi, \pi')\). The cost of a path \((\pi_1, \ldots, \pi_K)\) is defined as

\[c(\pi_1, \ldots, \pi_K) = \sum_{k=1}^{K-1} c(\pi_k, \pi_{k+1}). \]

One can then remark that Eq. (3) implies that \(c(\pi, \pi') = 0\) whenever \(\mathcal{F}_{\pi, \pi'} > 0\), that is whenever there is a positive probability to reach \(\pi'\) from \(\pi\) via the unperturbed process and that \(c(\pi, \pi')\) is bounded above by the number of distinct prices between \(\pi\) and \(\pi'\), that is \(c(\pi, \pi') \leq \delta(\pi, \pi')\) (let us remind that \(\delta(\pi, \pi') := \text{card} \{(i, j) \mid \pi_{ij} \neq \pi'_{ij}\}\)).
Let us then prove that \( \pi \) is stochastically stable. We shall precisely prove that
\[
r(\pi) > 3 \quad \text{and} \quad c(\pi, \pi') \leq 3.
\]

- Finally, the corollary of \( \pi \) is defined as the maximal cost of a transition to \( \pi' \)

\[
\sigma(\pi) = \max_{\pi \neq \pi} \min_{\pi' \in S(\pi, \pi')} c(s).
\]

Following Theorem 1 in Ellison (2000), in order to prove that \( \pi \) is stochastically stable, it suffices to prove that its radius is greater than its corollary. We shall precisely prove that

\[
r(\pi) > 3 \quad \text{and} \quad c(\pi, \pi') \leq 3.
\]

Let us first prove that \( c(\pi, \pi') \leq 3. \) Part (ii) of assumption (II) ensures that there always exists a zero cost path from any population to a uniform one where each agent uses the same price. So, without loss of generality, we can restrict attention to populations \( \pi \) such that for all \( (i, j), \pi_{ij} = p. \) Moreover as in Section 3.2, we can without loss of generality restrict attention to the cases where \( p \) is such that \( p_2 > p_1 > p_3 \) or \( p_1 > p_2 > p_3. \) In either cases, one has \( u_1(x_1(p)) < 1/2 \) and \( u_2(x_3(p)) < 1/2. \)

Let then \( \pi' \) be such that for all \( i, \pi'_{1i} = \bar{p} \) and for all \( j \neq 1, \pi'_{ij} = \pi_{ij} = p. \) One clearly has

\[
c(\pi, \pi') = 0.
\]

Finally, as it is clear that \( \mu(\pi', \bar{p}) = \max_{\pi \in P} \mu(\pi, p), \) one has using part (ii) of assumption (II) that

\[
c(\pi', \pi) = 0.
\]

Eqs. (10)–(12) eventually yield

\[
\sigma(\pi) \leq 3.
\]

This ends the first part of the proof.

Let us then prove that \( r(\pi) > 3: \)

- Assumption (II) clearly implies that \( \sigma(\pi) = 1, \) so that for any \( \pi \neq \pi, \) one has \( c(\pi, \pi) \neq 0. \) Hence \( r(\pi) > 0. \)

- Moreover Eq. (3) implies that for any \( \pi \) such that \( c(\pi, \pi) \leq 2, \) one has for any \( p \neq \bar{p}, \mu(\pi, p) = 0 \) as well as \( \sum_{p \neq \bar{p}} \mu(\pi, p) \leq 2. \) Let us consider \( (i, j) \in \pi \) such that \( \pi_{ij} \neq \bar{p} \) and \( \bar{p} \in \mathbb{S} \) with \( T_{\pi}(\bar{p}) > 0. \) Assumption (I) implies that \( \xi_{ij} = \omega_0. \) As moreover, one clearly does not have \( \mu(\pi, \pi_{ij}) = \max_{p \neq \bar{p}} \mu(\pi, p), \) assumption (II) in fact implies that for any \( (i, j) \neq (i', j'), I^{(i, j)} \{ (i, j) \rightarrow (i', j') \} = 0. \) Hence, any \( \pi' \) such that \( \mu(\pi, \pi') = 0 \) must satisfy \( \mu(\pi', p) = 0 \) as well as \( \sum_{p \neq \bar{p}} \mu(\pi', p) \leq 2. \) By recursion the same can be proven for any population \( \pi'' \) such that \( c(s) = 0 \) for some \( s \in S(\pi, \pi''). \) Moreover, part (ii) of assumption (II) ensures that any such \( \pi'' \) satisfies \( \sigma(\pi'') \leq 0. \) This clearly implies that any such \( \pi'' \) and in particular \( \pi \) belong to \( D(\pi). \) We have hence proven that \( r(\pi) > 3. \)

- Finally, if \( \pi \) is such that \( c(\pi, \pi) \leq 3, \) either one has for any \( p \neq \bar{p}, \mu(\pi, p) = 0 \) as well as \( \sum_{p \neq \bar{p}} \mu(\pi, p) \leq 3 \) and one can prove as in the preceding case that \( \pi \in D(\pi), \) or there exists \( p \neq \bar{p} \) such that \( \mu(\pi, p) = 1. \) Without loss of generality, we can then assume that for all \( i, \pi(i, i) = p \) and for all \( j \neq 1, \pi(i, j) = \bar{p}. \) Also, following remark 3.2, we can restrict attention to the cases where \( p \) such that \( p_2 > p_1 > p_3 \) or \( p_1 > p_2 > p_3. \) In either cases, one has \( u_1(x_1(p)) < 1/2 \) and \( u_2(x_3(p)) < 1/2, \) whereas one has respectively \( u_1(x_1(p)) = \frac{1}{2} \) and \( u_2(x_3(p)) = \frac{1}{2}. \) Assumption (II) then implies that for any \( \xi \in \mathbb{S} \) with \( T_{\pi}(\xi) > 0, \) one has \( I^{(\pi, \xi)} \{ (1, j) \rightarrow (1, \bar{1}) \} = 0 \) and \( I^{(\pi, \xi)} \{ (3, j) \rightarrow (3, 1) \} = 0, \) although one might have \( I^{(\pi, \xi)} \{ (j, j) \rightarrow (2, \bar{1}) \} > 0. \) This implies that for any \( \pi' \) such that \( c(\pi, \pi') = 0, \) one has \( \mu(\pi', p) = 1, \mu_2(\pi', p) < M, \) and \( \mu_3(\pi', p) < M, \) as well as \( \mu(\pi', \bar{p}) \geq M - 1. \) An immediate recursion shows the same is true for any \( \pi'' \) such that \( c(s) = 0 \) for some \( s \in S(\pi, \pi''). \)

Part (ii) of assumption (II) ensures that any such \( \pi'' \) satisfies \( \sigma(\pi'') > 0. \) This clearly implies that any such \( \pi'' \) and in particular \( \pi \) belongs to \( D(\pi). \) We have hence proven that \( r(\pi) > 3. \)

This ends the proof.

Basic asymptotic properties of ergodic Markov chains yield a straightforward interpretation of theorem 1: as the mutation rate tends towards zero, the frequency with which the system lies in equilibrium tends towards one.

4. Concluding remarks

We have hence shown that in a framework where trading is governed by private prices and strategic behavior, agents who update their private prices by imitation and random mutation will eventually adopt the equilibrium price (but for vanishingly small perturbations) and obtain their equilibrium allocation in the Scarf economy. Hence, we provide some analytical support to the result obtained in series of simulations performed by Gintis: Gintis (2007) and Gintis (2012). One should however take note that this result has only been obtained at the expense of a considerable simplification of Gintis’s model of market exchange. It is also the case that our results rely crucially on the fact that mutations are drawn from a uniform distribution and hence the remark in Fudenberg and Harris (1992) is particularly relevant in our setting and in view of the generalization of our results to a broader class of economies: “Intuitively, the likelihood that a Wiener process will be able to “swim upstream” \( k \) meters against a deterministic flow depends both on the distance \( k \) and on the strength of the flow, while the probability that a discrete-time system jumps \( k \) or more meters “over the flow” in a single period depends on \( k \) but not on the strength of the flow. This explains the differences in the generality of the models’ conclusions, and suggests that long-run behavior may depend on the precise form of the deterministic dynamics in any model with continuous sample paths”.

It is easy to “jump over the flow” in the Scarf economy because it exhibits very strong complementarities and hence it always is Pareto improving for a majority of agents to shift from a non-equilibrium to the equilibrium price (i.e. the corollary of the

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8 That is, agents of type 2 might adopt price \( p \) but not agents of type 1 and 3.
equilibrium price is small compared to its radius). One cannot expect that such strong complementarities will be present in an arbitrary economy. There, price dynamics might indeed have to “swim against the flow”: a possible strategy to show the stochastic stability of equilibrium in the general setting is to prove that at a non-equilibrium price, there always is a local price shift that is, with some probability, Pareto improving for a majority of agents. Then one would have to show that series of such local shifts can eventually lead to the equilibrium price. This would amount to show that the modified coradius of the equilibrium price (in the sense of Ellison, 2000) is small compared to its radius.

Extending our result to a broader class of economies would certainly also require to relax some of the constraints bearing on the dynamics (assumption (I) to (III) above). In this respect, one can certainly also require to relax some of the constraints.

| A strong constraint we currently impose is to have the same number of agents of every type. This condition can also be dispensed of, e.g. if one assumes that trade takes place among group of agents representing shares of the initial endowments proportional to the total supply (that is one would have to set $\mu_j$ specifically for each type as: $\mu_j(\pi, p) := N_j \min \left( \frac{\mu_1(\pi, p)}{N_1}, \frac{\mu_2(\pi, p)}{N_2}, \frac{\mu_3(\pi, p)}{N_3} \right)$ where $N_j$ is the number of agents of type $j$).
| More generally, we could allow for trading to take place among asymmetric groups provided that the performance of an “efficient” price (e.g an equilibrium price) is not hindered by the fact that the proportion of traders of each type using it is completely at odds with the proportions of the different types in the economy. In other words, the evaluation of the fitness of a price must be performed in conditions which are akin, as far as quantities supplied are concerned, to these that would prevail at the economy-wide level, not in artificial market conditions that could be created as artifacts of the imitation process.
| The first part of condition (I) implies that trade takes place at a certain price within a group of agents only if at least an agent of each type uses this price. We can easily relax this assumption and assume that any trade that is feasible and consistent with individual demands take place even though the group only contains a subset of types. Indeed, the last argument of the proof of Theorem 1 would then apply to the case where two agents shift to a non-equilibrium price. More generally, rather than restricting trades to subgroups and making the allocation a specific function of the agent’s private price, one could take a more axiomatic approach and assume that the allocation of goods satisfies some efficiency criteria, e.g. à la Nash (1950) or à la Kalai and Smorodinsky (1975), given the budget constraints implied by the agents private prices. This would preserve the Pareto superiority of the equilibrium allocation so that the proof shall proceed in this case as well.

Yet, as the constraints on the dynamics are relaxed, more paths become feasible and have to be taken into account when computing the radius and the coradius, which are maxima and minima over set of paths. Hence a major difficulty ahead if our results are to be generalized is that the problem will likely become more and more complex from the combinatorial point to view as assumptions are relaxed. It might be worth recalling here that the original results in Gintis (2007) were obtained via numerical simulations. Combinatorial difficulties might suggest that an efficient approach would be to use analytical results as founding stones for a theory whose fine details would be brushed up thanks to simulations. After all, the main concern of Scarf when he wrote Scarf (1960) was the calculability of equilibrium.

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**References**