An Efficient Algorithm for Solving Two-Player Games

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July 25, 2006

1 Introduction

Suppose a two-player normal form game $\mathcal{N}$ has a square matrix. Then it is easy to determine if there is a completely mixed Nash equilibrium for $\mathcal{N}$. This suggests a general way of finding many Nash equilibria: check for the completely mixed Nash of all square submatrices, and eliminate those for which some excluded row or column gives a higher payoff than the candidate.

Let us call the resulting Nash equilibria *regular*. It is clear that there are only a finite number of regular equilibria, and hence this process cannot locate all Nash equilibria of the game, if only because there may be connected sets of Nash equilibria.

By an *extreme point* of a connected set of Nash equilibria I mean a point of the set that is not interior to any line segment in the set. My hypothesis is that *every extreme point of a connected set of Nash equilibria is regular, and every regular equilibrium is the extreme point of a connected set of Nash equilibria*. I don't have a proof of this, but it works for every example I have tried. I give two examples below.

The proof may well be easy, because we can make the following informal argument. In the interior of a set of Nash equilibia, some inequality constraint is strict, and this must become an equality constraint at an extreme point. Conversely, if a regular equilibrium is not strict, then there is some direction in which an equality constraint becomes slack, an hence the regular equilibrium is an extreme point of a set of Nash equilibria (it cannot be an interior point of a set of Nash equilibria, or it would not be regular).

If this "Extreme Point Theorem" is true, it follows that the regular equilibria account for all the Nash payoffs of the game. This is because payoffs must be constant on a connected set of equilibria.

However, it is possible to recover all the Nash equilibria if the Extreme Point Theorem is true, at least if the game is appropriately generic. Let S be a maximal set of regular equilibria that give the same payoffs to all players. Then the convex hull of S is a maximal connected set of Nash equilibria.

Clearly, very simple algorithms allow all of the above calculations to be made. Moreover, these assertions may generalize to n-player games.

2 Example: Spence Signaling

I choose the Spence Signaling game (Spence 1973), exhibited in Figure 1 because it has a rather rich structure of Nash equilibria. There is a strict Nash equilibrium (NY,SU), where player 1 plays N at $1L$ and Y at $1H$, and player 2 plays S against Y and U against N. The payoffs are (52/3, 10). Also, there is two-dimensional connected set of Nash equilibria with payoffs (58/3, 25/3). This is the convex hull of three Nash equilibria. The first is (NN,SS), where player 1 always chooses N and player 2 always chooses S. The second is (NN,US), where player 1 always chooses N and player 2 chooses U against Y and S against N. The third is $((1/2)NY + (1/2)NN,SS)$.

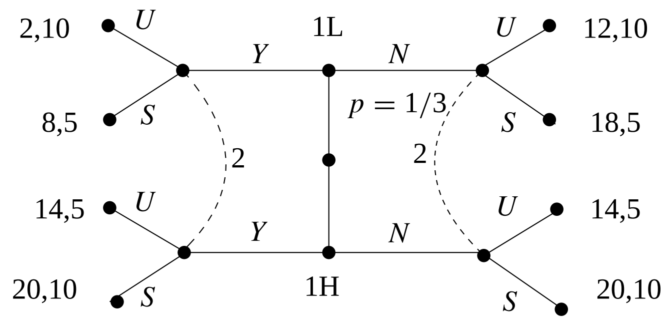

Figure 1: The Spence Signaling Game

The regular equilibria of this game are exactly these four extreme points: the strict equilibrium (NY,SU), and the three equilibria (NN,SS), (NN,US), and $((1/2)NY + (1/2)NN,SS)$.

3 Another Example

Consider the normal form matrix shown in Figure 2. The regular equilibria of this game are as folows, where ri means row i a cj means column j:

- r3,c4, with payoffs (2,3)
- r3,c3, with payoffs (2,3)
- r4,c3, with payoffs (2,2)
- r2,c3, with payoffs (2,2)
- r4,(1/2)c1+(1/2)c3, with payoffs (2,2)

3,2	3,2	1,1	1,1
2,2	0,1	2,2	0,1
0,0	0,0	2,3	2,3
2,2	0,1	2,2	0,1

Figure 2: A Normal form game with a complex Nash equilibrium structure

- r2,(1/2)c1+(1/2)c3, with payoffs (2,2)
- r1,c2, with payoffs (3,2)
- r1,c1, with payoffs (3,2)
- (3/4)r1+(1/4)r3,(1/4)c2+(3/4)c4 with payoffs (3/2,3/2)
- (3/4)r1+(1/4)r3,(1/4)c2+(3/4)c3 with payoffs (3/2,3/2)
- (3/4)r1+(1/4)r3,(1/4)c1+(3/4)c4 with payoffs (3/2,3/2)
- (3/4)r1+(1/4)r3,(1/4)c2+(1/2)c3+(1/4)c4 with payoffs (3/2,3/2)

Note that the first two r3,c4 and r3,c3 are the endpoints of an interval of Nash equilibria with payoffs (2,3). The next four regular equilibria are the extreme points of a two-dimensional convex set of Nash equilibria of the form

$$\alpha \mathrm{r4} + (1-\alpha)\mathrm{r2},\, \beta \mathrm{c1} + (1-\beta)\mathrm{c2} \quad \text{for } 0 \le \alpha \le 1,\, 0 \le \beta \le 1/2,$$

all with payoffs (2,2). The next two regular equilibria, r1,c2 and r1,c1, are again endpoints of an interval of Nash equilibria, with payoffs (3,2). Finally the last four are extreme points of a two-dimensional convex set of Nash equilibria of the form

$$(3/4)\mathrm{r4}+(1/4)\mathrm{r3},\, \alpha \mathrm{c1}+(1-\alpha)\mathrm{c2}+\beta \mathrm{c3}+(1-\beta)\mathrm{c4} \quad \text{for } 0 \le \alpha \le 1/4,\, 1/2 \le \beta \le 3/4,$$

with payoffs (3/2,3,2).

4 Conclusion

What is to be proved? The Extreme Point Theorem.

Actually, there is a slight complication. The algorithm for finding the completely mixed equilibria of a square normal form game can, on a set of measure zero, assign zero probability to one or more pure strategies; i.e., we can invert the appropriate matrix, but one or more of the solution entries can be zero.

References

Spence, A. Michael, "Job Market Signaling," *Quarterly Journal of Economics* 90 (1973):225–243.

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