5

Pure-Strategy Nash Equilibria

> Competition among agents . . . has merit solely as a device to extract information optimally. Competition per se is worthless.
>
> Bengt Holmström

A *pure strategy* Nash equilibrium of a game is a Nash equilibrium in which each player uses a pure strategy, but not necessarily one determined by the iterated elimination of dominated strategies (§4.1). Not every game has a pure-strategy Nash equilibrium. Indeed, there are even very simple 2×2 normal form games with no pure-strategy Nash equilibria—for instance throwing fingers (§3.8), where the Nash equilibrium consists in each player throwing one or two fingers, each with probability 1/2.

This chapter explores some of the more interesting applications of games with pure-strategy equilibria. As you will see, we obtain extremely deep results in various branches of economic theory, including altruism (§5.16,§5.17), the tragedy of the commons (§5.5), the existence of pure-strategy equilibria in games of perfect information (§5.6), the real meaning of competition (it is probably not what you think) (§5.3), honest signaling equilibria (§5.9) and (§5.18). Another feature of this chapter is its use of agent-based modeling, in no-draw, high-low poker (§5.7), to give you a feel for the dynamic properties of games for which the Nash equilibrium concept is a plausible description of reality.

5.1 Price Matching as Tacit Collusion

Bernie and Manny both sell DVD players and both have unit costs of 250. They compete on price: the low-price seller gets all the market and they split the market if they have equal prices. Explain why the only Nash equilibrium has both firms charging 250, splitting the market and making zero profit.

Suppose that the monopoly price for DVD players (the price that maximizes the sum of the profits of both firms) is 300. Now suppose Bernie advertises that if a customer buys a DVD player from him for 300 and discovers he or she can buy it cheaper at Manny's, Bernie will refund the full purchase price. Suppose Manny does the same thing. Show that it is now Nash for both stores to charge 300. Conclusion: pricing strategies that seem to be supercompetitive can in fact be anticompetitive!

5.2 Competition on Main Street

The residents of Pleasantville live on Main Street, which is the only road in town. Two residents decide to set up general stores. Each can locate at any point between the beginning of Main Street, which we will label 0, and the end, which we will label 1. The two decide independently where to locate and they must remain there forever (both can occupy the same location). Each store will attract the customers who are closest to it and the stores will share equally customers who are equidistant between the two. Thus, for instance, if one store locates at point x and the second at point $y > x$, then the first will get a share $x + (y - x)/2$ and the second will get a share $(1 - y) + (y - x)/2$ of the customers each day (draw a picture to help you see why). Each customer contributes \$1.00 in profits each day to the general store it visits.

a. Define the actions, strategies, and daily payoffs to this game. Show that the unique pure-strategy Nash equilibrium where both players locate at the midpoint of Main Street;
b. Suppose there are three General Stores, each independently choosing a location point along the road (if they all choose the same point, two of them share a building). Show that there is no pure-strategy Nash equilibrium. Hint: First show that there is no pure-strategy Nash equilibrium where all three stores locate on one half of Main Street. Suppose two stores locate on the left half of Main Street. Then, the third store should locate a little bit to the right of the rightmost of the other two stores. But, then the other two stores are not best responses. Therefore, the assumption is false. Now finish the proof.

5.3 Markets as Disciplining Devices: Allied Widgets

In *The Communist Manifesto* of 1848, Karl Marx offered a critique of the nascent capitalist order that was to resound around the world and fire the imagination of socialists for nearly a century and a half.

> The bourgeoisie, wherever it has got the upper hand, has put an end to all feudal, patriarchal, idyllic relations. It has pitilessly torn asunder the motley feudal ties that bound man to his "natural superiors", and has left no other nexus between man and man than naked self-interest, than callous "cash payment". . . . It has resolved personal worth into exchange value, and in place of the numberless indefeasible chartered freedoms, has set up that single, unconscionable freedom: Free Trade. (Marx 1948)

Marx's indictment covered the two major institutions of capitalism: market competition and private ownership of businesses. Traditional economic theory held that the role of competition was to set prices, so supply equals demand. If this were correct, a socialist society that took over ownership of the businesses could replace competition by a central-planning board that sets prices using statistical techniques to assess supply and demand curves.

The problem with this defense of socialism is that traditional economic theory is wrong. The function of competition is to reveal private information concerning the shape of production functions and the effort of the firm's managers and use that information to reward hard work and the efficient use of resources by the firm. Friedrich von Hayek (1945) recognized this error and placed informational issues at the heart of his theory of capitalist competition. By contrast, Joseph Schumpeter (1942), always the bitter opponent of socialism, stuck to the traditional theory and predicted the inevitable victory of the system he so hated (Gintis 1991).

This problem pins down analytically the notion that competition is valuable because it reveals otherwise private information. In effect, under the proper circumstances, market competition subjects firms to a prisoner's dilemma in which it is in the interest of each producer to supply high effort, even in cases where consumers and the planner cannot observe or contract for effort itself. This is the meaning of Bengt Holmström's quotation at the head of this chapter.

If Holmström is right, and both game-theoretic modeling and practical experience indicate that he is, the defense of competitive markets in neoclassical economics is a great intellectual irony. Because of Adam Smith, supporters of the market system have defended markets on the grounds that they allocate goods and services efficiently. However, empirical estimates of the losses from monopoly, tariffs, quotas, and the like indicate that misallocation has little effect on per capita income or the rate of economic growth (Hines 1999). By contrast, the real benefits of competition, which include its ability to turn private into public information, have come to light only through game-theoretic analysis. The following problem is a fine example of such analysis.

Allied Widgets has two possible constant returns to scale production techniques: fission and fusion. For each technique, Nature decides in each period whether marginal cost is 1 or 2. With probability $\theta \in (0, 1)$, marginal cost is 1. Thus, if fission is high cost in a given production period, the manager can use fusion, which will be low cost with probability θ. However, it is costly for the manager to inspect the state of Nature and if he fails to inspect, he will miss the opportunity to try fusion if the cost of fission is high.

Allied's owner cannot tell whether the manager inspected or not, but he does know the resulting marginal cost and can use this to give an incentive wage to the manager. Figure 5.1 shows the manager's decision tree, which assumes the manager is paid a wage w_1 when marginal costs are low and w_2 when marginal costs are high, the cost of inspecting is α and the manager has a logarithmic utility function over income: $u(w) = \ln w$.[1]

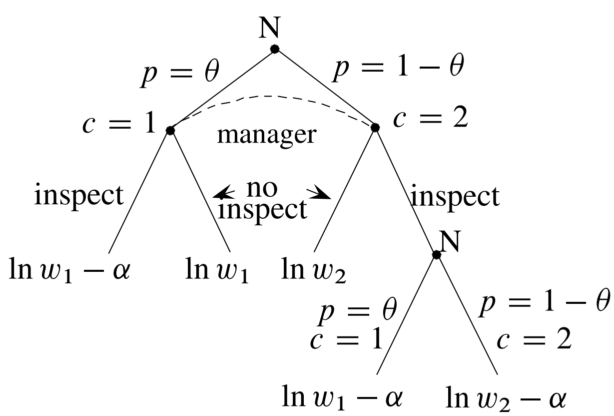

Figure 5.1. The Allied Widgets problem

To induce the manager to inspect the fission process, the owner decides to pay the manager a wage w_1 if marginal cost is low and $w_2 < w_1$ if

[1] The logarithmic utility function is a reasonable choice, because it implies constant relative risk aversion; that is, the fraction of wealth an agent desires to put in a particular risky security is independent of wealth.

marginal cost is high. But how should the owner choose w_1 and w_2 to maximize profits? Suppose the manager's payoff is $\ln w$ if he does not inspect, $\ln w - \alpha$ if he inspects and $\ln w_o$ if he does not take the job at all. In this case, w_o is called the manager's *reservation wage* or *fallback position*.

The expression that must be satisfied for a wage pair (w_1, w_2) to induce the manager to inspect the fission process is called the *incentive compatibility constraint*. To find this expression, note that the probability of using a low-cost technique if the manager does not inspect is θ, so the payoff to the manager from not inspecting (by the expected utility principle) is

$$\theta \ln w_1 + (1-\theta) \ln w_2.$$

If the manager inspects, both techniques will turn out to be high cost with probability $(1-\theta)^2$, so the probability that at least one of the techniques is low cost is $1-(1-\theta)^2$. Thus, the payoff to the manager from inspecting (again by the expected utility principle) is

$$[1-(1-\theta)^2] \ln w_1 + (1-\theta)^2 \ln w_2 - \alpha.$$

The incentive compatibility constraint is then

$$\theta \ln w_1 + (1-\theta) \ln w_2 \leq [1-(1-\theta)^2] \ln w_1 + (1-\theta)^2 \ln w_2 - \alpha.$$

Because there is no reason to pay the manager more than absolutely necessary to get him to inspect, we can assume this is an equality,[2] in which case the constraint reduces to $\theta(1-\theta)\ln[w_1/w_2] = \alpha$, or

$$w_1 = w_2 e^{\frac{\alpha}{\theta(1-\theta)}}.$$

For instance, suppose $\alpha = 0.4$ and $\theta = 0.8$. Then $w_1 = 12.18 w_2$; that is, the manager must be paid more than twelve times as much in the good state as in the bad!

But the owner must also pay the manager enough so that taking the job, compared to taking the fallback w_o is worthwhile. The expression that must be satisfied for a wage pair (w_1, w_2) to induce the manager to take the job is

[2]Actually, this point may not be obvious and is false in the case of repeated principal-agent models. This remark applies also to our assumption that the participation constraint, defined in the text, is satisfied as an equality.

called the *participation constraint*. In our case, the participation constraint is

$$[1-(1-\theta)^2]\ln w_1 + (1-\theta)^2 \ln w_2 - \alpha \geq \ln w_o.$$

If we assume that this is an equality and using the incentive compatibility constraint, we find $w_o = w_2 e^{\alpha/(1-\theta)}$, so

$$w_2 = w_o e^{-\frac{\alpha}{(1-\theta)}}, \qquad w_1 = w_o e^{\frac{\alpha}{\theta}}.$$

Using the above illustrative numbers and if we assume $w_o = 1$, we get

$$w_2 = 0.14, \qquad w_1 = 1.65.$$

The expected cost of the managerial incentives to the owner is

$$[1-(1-\theta)^2]w_1 + (1-\theta)^2 w_2 = w_o\left[\theta(2-\theta)e^{\frac{\alpha}{\theta}} + (1-\theta)^2 e^{-\frac{\alpha}{(1-\theta)}}\right].$$

Again, using our illustrative numbers, we get expected cost

$$0.96(1.65) + 0.04(0.14) = 1.59.$$

So where does competition come in? Suppose Allied has a competitor, Axis Widgets, subject to the same conditions of production. In particular, whatever marginal-cost structure Nature imposes on Allied, Nature also imposes on Axis. Suppose also that the managers in the two firms cannot collude. We can show that Allied's owner can write a Pareto-efficient contract for the manager using Axis's marginal cost as a signal, satisfying both the participation and incentive compatibility constraints and thereby increasing profits. They can do this by providing incentives that subject the managers to a prisoner's dilemma, in which the dominant strategy is to defect, which in this case means to inspect fission in search of a low-cost production process.

To see this, consider the following payment scheme, used by both the Axis and the Allied owners, where $\phi = 1 - \theta + \theta^2$, which is the probability that both managers choose equal-cost technologies when one manager inspects and the other does not (or, in other words, one minus the probability that the first choice is high and the second low). Moreover, we specify the parameters β and γ so that $\gamma < -\alpha(1-\theta+\theta^2)/\theta(1-\theta)$ and $\beta > \alpha(2-\phi)/(1-\phi)$. This gives rise to the payoffs to the manager shown in the table, where the example uses $\alpha = 0.4$, $\theta = 0.8$, and $w_0 = 1$.

Allied Cost	Axis Cost	Allied Wage	Example
$c = 1$	$c = 1$	$w^* = w_o e^{\alpha}$	$w^* = 1.49$
$c = 2$	$c = 2$	$w^* = w_o e^{\alpha}$	$w^* = 1.49$
$c = 1$	$c = 2$	$w^+ = w_o e^{\beta}$	$w^+ = 54.60$
$c = 2$	$c = 1$	$w^- = w_o e^{\gamma}$	$w^- = 0.02$

We will show that the manager will always inspect and the owner's expected wage payment is w^*, which merely pays the manager the equivalent of the fallback wage. Here is the normal form for the game between the two managers.

	Inspect	Shirk
Inspect	$\ln w^* - \alpha$ $\ln w^* - \alpha$	$\phi \ln w^* + (1-\phi)\ln w^+ - \alpha$ $\phi \ln w^* + (1-\phi)\ln w^-$
Shirk	$\phi \ln w^* + (1-\phi)\ln w^-$ $\phi \ln w^* + (1-\phi)\ln w^+ - \alpha$	$\ln w^*$ $\ln w^*$

Why is this so? The inspect/inspect and shirk/shirk entries are obvious. For the inspect/shirk box, with probability ϕ the two managers have the same costs, so they each get $\ln w^*$ and with probability $1-\phi$ the Allied manager has low costs and the Axis manager has high costs, so the former gets $\ln w^+$ and the latter gets $\ln w^-$.

To show that this is a prisoner's dilemma, we need only show that

$$\ln w^* - \alpha > \phi \ln w^* + (1-\phi)\ln w^-$$

and

$$\phi \ln w^* + (1-\phi)\ln w^+ - \alpha > \ln w^*.$$

The first of these becomes

$$\ln w_o > \phi \ln w_o + \phi\alpha + (1-\phi)\ln w_o + (1-\phi)\gamma,$$

or $\gamma < -\phi\alpha/(1-\phi)$, which is true by assumption. The second becomes

$$\ln w^+ > \frac{\alpha}{1-\phi} + \ln w^*.$$

or $\beta > \alpha\frac{2-\phi}{1-\phi}$, which is also true by assumption.

Note that in our numerical example the cost to the owner is $w^* = 1.49$ and the incentives for the managers are given by the normal form matrix

	Inspect	Shirk
Inspect	0,0	0.58,-0.30
Shirk	$-0.30, 0.58$	0.4,0.4

This example shows that markets may be disciplining devices in the sense that they reduce the cost involved in providing the incentives for agents to act in the interests of their employers or clients, even where enforceable contracts cannot be written. In this case, there can be no enforceable contract for managerial inspecting. Note that in this example, even though managers are risk averse, imposing a structure of competition between the managers means each inspects and the cost of incentives is no greater than if a fully specified and enforceable contract for inspecting could be written.

Of course, if we weaken some of the assumptions, Pareto-optimality will no longer be attainable. For instance, suppose when a technique is low cost for one firm, it is not necessarily low cost for the other, but rather is low cost with probability $q > 1/2$. Then competition between managers has an element of uncertainty and optimal contracts will expose the managers to a positive level of risk, so their expected payoff must be greater than their fallback.

5.4 The Tobacco Market

The demand for tobacco is given by

$$q = 100000(10 - p),$$

where p is the price per pound. However, there is a government price support program for tobacco that ensures that the price cannot go under \$0.25 per pound. Three tobacco farmers have each harvested 600,000 pounds of tobacco. Each must make an independent decision on how much to ship to the market and how much to discard.

a. Show that there are two Nash equilibria, one in which each farmer ships the whole crop and a second in which each farmer ships 250,000 pounds and discards 350,000 pounds.
b. Are there any other Nash equilibria?

5.5 The Klingons and the Snarks

Two Klingons are eating from a communal cauldron of snarks. There are 1,000 snarks in the cauldron and the Klingons decide individually the rate r_i, $(i = 1, 2)$ at which they eat per eon. The net utility from eating snarks, which depends on both the amount eaten and the rate of consumption (too slow depletes the Klingon Reservoir, too fast overloads the Klingon Kishkes) is given by

$$u_i = 4q_i + 50r_i - r_i^2,$$

where q_i is the total number of snarks Klingon i eats. Since the two Klingons eventually eat all the snarks, $q_i = 1000r_i/(r_1 + r_2)$.

a. If they could agree on an optimal (and equal) rate of consumption, what would that rate be?
b. When they choose independently, what rate will they choose?
c. This problem illustrates the tragedy of the commons (Hardin 1968), in which a community (in this case the two Klingons, though it usually involves a larger number of individuals) overexploits a resource (in this case the bowl of snarks) because its members cannot control access to the resource. Some economists believe the answer is simple: the problem arises because no one owns the resource. So give an individual the right to control access to the resource and let that individual sell the right to extract resources at a rate r to the users. To see this, suppose the cauldron of snarks is given to a third Master Klingon and suppose the Master Klingon charges a diner a fixed number of drecks (the Klingon monetary unit), chosen to maximize his profits, for the right to consume half the cauldron. Show that this will lead to an optimal rate of consumption.

This "create property rights in the resource" solution is not always satisfactory, however. First, it makes the new owner rich and everyone else poor. This could possibly be solved by obliging the new owner to pay the community for the right to control the resource. Second, it may not be possible to write a contract for the rate of resource use; the community as a whole may be better at controlling resource use than a single owner (Ostrom, Walker, and Gardner 1992). Third, if there is unequal ability to pay among community members, the private property solution may lead to an unequal distribution of resources among community members.

5.6 Chess: The Trivial Pastime

A *finite game* is a game with a finite number of nodes in its game tree. A game of *perfect information* is a game where every information set is a single node and Nature has no moves. In 1913 the famous mathematician Ernst Zermelo proved that in chess either the first mover has a winning pure strategy, the second mover has a winning pure strategy, or either player can force a draw. This proof was generalized by Harold Kuhn (1953), who proved that every finite game of perfect information has a pure-strategy Nash equilibrium. In this problem you are asked to prove a special case of this, the game of chess.

Chess is clearly a game of perfect information. It is also a finite game, because one of the rules is that if the board configuration is repeated three times, the game is a draw. Show that in chess, either Black has a winning strategy, or White has a winning strategy, or both players have strategies that can force a draw.

Of course, just because there exists an optimal strategy does not imply that there is a feasible way to find one. There are about 10^{47} legal positions in chess, give or take a few orders of magnitude, implying a game-tree complexity that is almost the cube of this number. This is far more than the number of atoms in the universe.

5.7 No-Draw, High-Low Poker

Alice and Bob are playing cards. The deck of cards has only two types of card in equal numbers: high and low. Each player each puts 1 in the pot. Alice is dealt a card (by Nature). After viewing the card, which Bob cannot see, she either raises or stays. Bob can stay or fold if Alice raises and can raise or stay if Alice stays. If Alice raises, she puts an additional 1 in the pot. If Bob responds by folding, he looses and if he responds by staying, he must put an additional 1 in the pot. If Alice stays and Bob raises, both must put an additional 1 in the pot. If the game ends without Bob folding, Alice wins the pot if she has a high card and loses the pot if she has a low card. Each player's objective is to maximize the expected value of his or her winnings. The game tree is in figure 5.2

We now define strategies for each of the players. Alice has two information sets (each one a node) and two choices at each. This gives four strategies, which we label RR, RS, SR, and SS. These mean "raise no