# 11

# Dynamical Systems

History is Spirit at war with itself.

Georg Wilhelm Freidrich Hegel

We have studied Nash equilibria of games, but *do* games reach Nash equilibrium and, if so, by what process? If there are several Nash equilibria, to which one does the game go? Indeed, what are Nash equilibria equilibria of? To answer these questions we will study the behavior of *dynamical systems* that are generally not in equilibrium but which, under appropriate conditions, approach a state of equilibrium over time, or orbit equilibria the way planets orbit the sun, or have some other love-hate relationship with equilibria (e.g., strange attractors).

We can apply several analytical tools in treating strategic interactions as dynamical systems, including difference equations, stochastic processes (such as Markov chains and diffusion processes), statistical mechanics, and differential equations. The differential equation approach is the most basic and has the quickest payoff, so that is what we will develop in this chapter.

## 11.1 Dynamical Systems: Definition

Suppose $x = (x_1, \ldots, x_n)$ is a point in $n$-dimensional space $\mathbf{R}^n$ that traces out a curve through time. We can describe this as

$$x = x(t) = (x_1(t), \ldots, x_n(t)) \qquad \text{for } -\infty < t < \infty.$$

Often we do not know $x(t)$ directly, but we do know the forces determining its rate and direction of change in some region of $\mathbf{R}^n$. We thus have

$$\dot{\mathbf{x}} = f(\mathbf{x}) \qquad \mathbf{x} \in \mathbf{R}^n, \tag{11.1}$$

where the "dot" indicates the derivative with respect to $t$, so $\dot{\mathbf{x}} = dx/dt$. We always assume $f$ has continuous partial derivatives. If we write these vector equations out in full, we get

$$\begin{aligned}
\frac{dx_1}{dt} &= f^1(x_1, \ldots, x_n), \\
\frac{dx_2}{dt} &= f^2(x_1, \ldots, x_n), \\
&\vdots \\
\frac{dx_n}{dt} &= f^n(x_1, \ldots, x_n),
\end{aligned}$$

We call this a set of *first-order ordinary differential equations* in $n$ unknowns. It is "first-order" because no derivative higher than the first appears. It is "ordinary" as opposed to "partial" because we want to solve for a function of the single variable $t$, as opposed to solving for a function of several variables.

We call $\mathbf{x}(t)$ a *dynamical system* if it satisfies such a set of ordinary differential equations, in the sense that $\dot{\mathbf{x}}(t) = f(\mathbf{x}(t))$ for $t$ in some (possibly infinite) interval. A *fixed point*, also called a *critical point*, or a *stationary point*, is a point $\mathbf{x}^* \in \mathbf{R}^n$ for which $f(\mathbf{x}^*) = 0$.

## 11.2 Population Growth

Suppose the rate of growth of fish in a lake is $r$. Then the number $y$ of fish in the lake is governed by the equation

$$\dot{y} = ry.$$

We can solve this equation by "separation of variables," bringing all the expressions involving $t$ on the right, and all the expressions involving $y$ on the left. This is not possible for just any differential equation, of course, but it is possible in this case. This gives

$$\frac{dy}{y} = r\,dt.$$

Now we integrate both sides, getting $\ln y = rt + a$, where $a$ is a constant of integration. Taking the antilogarithm of both sides, we get

$$y = be^{rt},$$

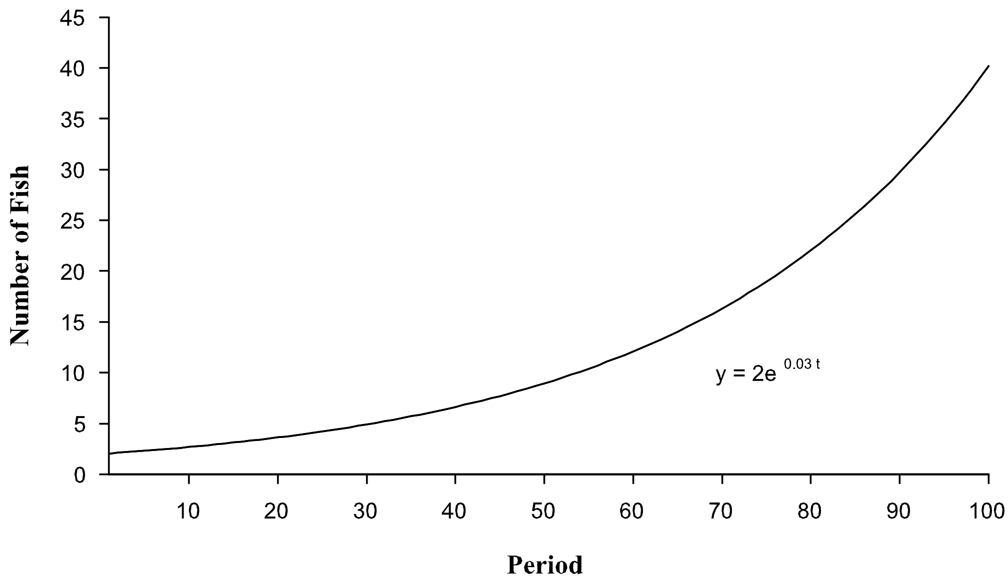


Figure 11.1. The exponential growth of fish in lake. The initial population is $y_o = 2$, and the rate of growth is $r = 3.0\%$.

where $b = e^a$ is another constant of integration.

We determine the constant of integration by noting that if the number of the fish in the lake at time $t = 0$ is $y_o$, then we must have $b = y_o$. This gives the final solution

$$y = y_o e^{rt}. \tag{11.2}$$

This function is graphed in figure 11.1.

## 11.3 Population Growth with Limited Carrying Capacity

Equation (11.2) predicts that the fish population can grow without bounds. More realistically, suppose that the more fish, the lower the rate of growth of fish. Let $\eta$ be the "carrying capacity" of the lake—the number of fish such that the rate of growth of the fish population is zero. The simplest expression for the growth rate of the fish population, given that the growth rate is $r$ when $y$ is near zero, is then $r(1 - y/\eta)$. Our differential equation then becomes

$$\dot{y} = r\left(1 - \frac{y}{\eta}\right) y \qquad \eta, r > 0. \tag{11.3}$$

Note that the dynamical system given by this equation has two fixed points: $y^* = 0$, where the fish population is zero, and $y^* = \eta$, where the population is just equal to the carrying capacity.

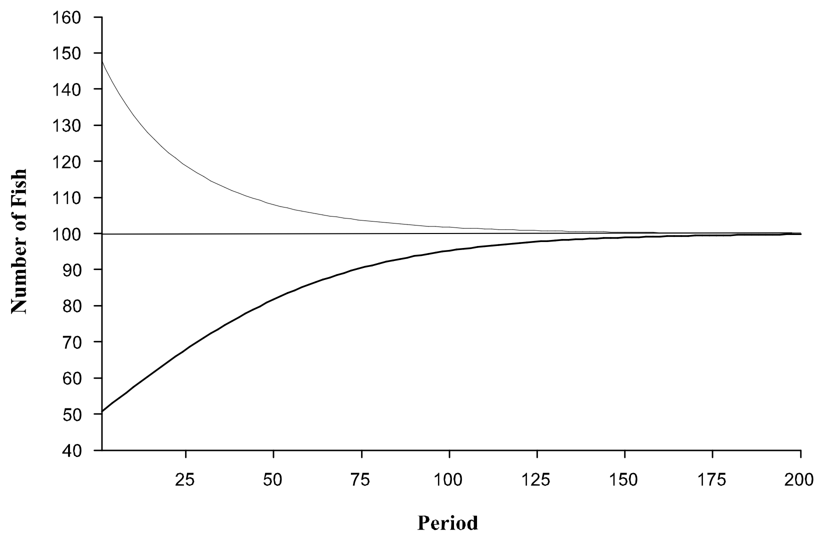


Figure 11.2. Population growth with limited carrying capacity

To solve the equation, we separate variables, getting

$$\frac{dy}{y(\eta - y)} = \frac{r}{\eta} dt.$$

We now integrate both sides, getting

$$\int \frac{dy}{y(\eta - y)} = \frac{r}{\eta} t + a, \tag{11.4}$$

where $a$ is a constant of integration. We use the method of partial fractions to write

$$\frac{1}{y(\eta - y)} = \frac{1}{\eta}\left[\frac{1}{\eta - y} + \frac{1}{y}\right].$$

Thus, we have

$$\int \frac{dy}{y(\eta - y)} = \frac{1}{\eta}\left[\int \frac{dy}{\eta - y} + \int \frac{dy}{y}\right]$$

$$= \frac{1}{\eta} \ln \frac{y}{\eta - y}.$$

Substituting into (11.4), we get

$$\ln \frac{y}{\eta - y} = rt + a\eta.$$

Taking antilogarithms of both sides, this becomes

$$\frac{y}{\eta - y} = be^{rt},$$

where $b = e^{a\eta}$ is another constant of integration. If the number of fish in the lake at time $t = 0$ is $y_o$, then we must have $b = y_o/(\eta - y_o)$, which can be either positive or negative, depending on whether the initial fish population is larger or smaller than the stationary population size $\eta$.

Now we can solve this equation for $y$, getting

$$y = \frac{\eta}{Ae^{-rt} + 1},$$

where $A = (\eta - y_o)/y_o$. Note that this equation predicts a smooth movement from disequilibrium to stationarity as $t \to \infty$. A picture of the process is given in figure 11.2.

## 11.4 The Lotka-Volterra Predator-Prey Model

Foxes eat rabbits. Suppose we normalize the rabbit population at a point in time to a fraction $x$ of its maximum, given the carrying capacity of its habitat when foxes are absent, and suppose the fox population at a point in time is a fraction $y$ of its maximum, given the carrying capacity of its habitat when there is an unlimited supply of rabbits. Suppose foxes are born at the rate $\delta_1 x$ but die at the rate $\gamma_1(1-x)$. We then have $\dot{y}/y = \delta_1 x - \gamma_1(1-x)$, which we can write as

$$\dot{y} = \delta y(x - \gamma), \qquad \delta > 0, 1 > \gamma > 0, \tag{11.5}$$

where we have written $\delta = \delta_1 + \gamma_1$ and $\gamma = \gamma_1/(\delta_1 + \gamma_1)$. Equation (11.5) expresses the rate of growth $\dot{y}/y$ as a function of the frequency of rabbits.

Suppose the natural rate of growth of rabbits is $g > 0$, but predation reduces the rate of growth by $\mu y$, so

$$\dot{x} = x(g - \mu y). \tag{11.6}$$

Now, (11.5) and (11.6) form a pair of differential equations in two unknowns ($x$ and $y$), the solution to which is a dynamical system known as the *Lotka-Volterra predator-prey model.*

How do we solve this equation? There is no solution in closed form (e.g., using polynomials, trigonometric functions, logarithms, and exponentials). We can, however, discover the properties of such equations without solving them explicitly.

We begin such an analysis with a *phase diagram* of the differential equations. The phase diagram for the Lotka-Volterra model is depicted in figure 11.3.

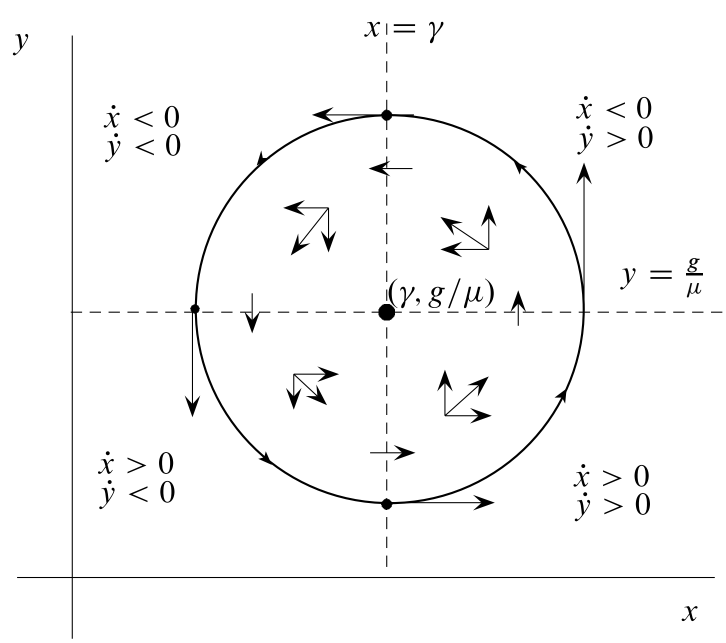


Figure 11.3. Phase diagram of Lotka-Volterra system

The horizontal dotted line represents the condition $dx/dt = 0$, and the vertical dotted line represents the condition $dy/dt = 0$. The fixed point is at $(\gamma, g/\mu)$, where the two intersect. The little arrows show in which direction the flow of the dynamical system moves for that particular point

$(x, y)$. The arrows point northward when $dy/dt > 0$ and southward when $dy/dt < 0$, and they point eastward when $dx/dt > 0$ and westward when $dx/dt < 0$. The arrows are vertical where $dx/dt = 0$ because motion is purely north-south instantaneously at such a point, and are horizontal where $dy/dt = 0$, because motion is purely east-west at such a point. In each of the four quadrants marked off by the dotted lines, the direction of the flow is qualitatively similar. Thus to the northeast of the fixed point, the flow is northwest; to the northwest, the flow is southwest; to the southwest, the flow is southeast; and to the southeast of the fixed point, the flow is to the northeast. So it is clear that the flow circles counterclockwise about the fixed point. However, we cannot tell *a priori* whether the flow circles into the fixed point, circles outward to infinity, or forms closed circuits about the fixed point.

To show that the Lotka-Volterra has closed orbits (§11.5), we find a function that is constant on any trajectory of the dynamical system and show that this function is monotonic (strictly increasing or decreasing) along a ray starting from the fixed point and pointing northeast.[1]

Suppose we have such a function $f$ and consider a path starting at a point $\mathbf{x}$ on the ray, making one complete revolution around the fixed point and hitting the ray again, say at $\mathbf{y}$. Because $f$ is constant on the path, $f(\mathbf{x}) = f(\mathbf{y})$. But because $f$ is monotonic on the ray, we must have $\mathbf{x} = \mathbf{y}$, so the path is a closed orbit (§11.5). First, we eliminate $t$ from the equations for $\dot{x}$ and $\dot{y}$ by dividing the first by the second, getting

$$\frac{dy}{dx} = \frac{\delta y(x-\gamma)}{x(g-\mu y)}.$$

Now we separate variables, pulling all the $x$'s to the right, and all the $y$'s to the left:

$$\frac{g-\mu y}{y}dy = \frac{\delta(x-\gamma)}{x}dx.$$

Now we integrate both sides, getting

$$g\ln y - \mu y = \delta x - \delta\gamma \ln x + C,$$

[1]We say a function $f(x)$ is (1) *increasing* if $x > y$ implies $f(x) \geq f(y)$; (2) *strictly increasing* if $x > y$ implies $f(x) > f(y)$; (3) *decreasing* if $x > y$ implies $f(x) \leq f(y)$; and (4) *strictly decreasing* if $x > y$ implies $f(x) < f(y)$.

where $C$ is an arbitrary constant of integration. Bringing all the variables over to the left and taking the antilogarithm, we get

$$y^g x^{\delta\gamma} e^{-(\mu y+\delta x)} = e^C. \tag{11.7}$$

So now we have an expression that is constant along any trajectory of the Lotka-Volterra dynamical system.

Now, consider a ray $(x, y)$ that starts at the fixed point $(\gamma, g/\mu)$ and moves to the northeast in a direction heading away from the origin. We can write this as $x = \gamma s$, $y = (g/\mu)s$, where $s$ is a parameter measuring the distance from the fixed point. Note that when $s = 1$, $(x, y)$ is at the fixed point. Substituting in (11.7), we get

$$\left(\frac{g}{\mu}\right)^g \gamma^{\delta\gamma} s^{g+\delta\gamma} e^{-(g+\delta\gamma)s} = e^C.$$

This looks forbidding, but it's really not. We pull the first two terms on the left over to the right, and then take the $(g + \delta\gamma)$-th root of both sides. The right-hand side is a complicated constant, which we can abbreviate by $D$, and the left is just $se^{-s}$, so we have

$$se^{-s} = D. \tag{11.8}$$

If we can show that the left-hand side is strictly decreasing for $s > 1$, we are done, because then any $s > 1$ that satisfies (11.8) must be unique. We take the derivative of the left-hand side, getting

$$e^{-s} - se^{-s} = (1 - s)e^{-s},$$

which is negative for $s > 1$. This shows that the dynamical system moves in closed orbits (§11.5) around the fixed point.

It follows from this analysis that if the system begins out of equilibrium, both the fraction of rabbits and foxes will go through constant-amplitude oscillations around their equilibrium values forever. We shall later characterize this as an *asymmetric evolutionary game* (§12.17) for which this oscillatory behavior is quite typical.

## 11.5 Dynamical Systems Theory

With these examples under our belt, we can address the basic theory of dynamical systems (a.k.a. differential equations).[2]

Suppose a dynamical system is at a point $\mathbf{x}_o$ at time $t_o$. We call the locus of points through which the system passes as $t \to \infty$ the *forward trajectory* of the system through $\mathbf{x}_o$, or the *trajectory* of the system starting at $\mathbf{x}_o$. The *backward trajectory* of the system through $\mathbf{x}_o$ is the locus of points through which the system passes as $t \to -\infty$. The forward and backward trajectories are together called the *trajectory* through $\mathbf{x}_o$.

Clearly if a dynamical system is at a fixed point $\mathbf{x}^*$, it will stay there forever, so the trajectory starting at $\mathbf{x}^*$ is simply $\mathbf{x}^*$ itself. However, if we perturb the system a little from $\mathbf{x}^*$ by choosing a new initial point $\mathbf{x}_o$ at time $t = 0$, there are several things that can happen. We begin with a couple of definitions. If $\mathbf{x} \in \mathbf{R}^n$, and $r > 0$, we define a *ball of radius $r$* around $\mathbf{x}$, which we write $B_r(\mathbf{x})$, as the set of points $\mathbf{y} \in \mathbf{R}^n$ whose distance from $\mathbf{x}$ is less than $r$. We define a *neighborhood* of $\mathbf{x}$ to be any subset of $\mathbf{R}^n$ that contains some ball around $\mathbf{x}$. Finally, we say a set in $\mathbf{R}^n$ is an *open set* if it is a neighborhood of each of its points. Note that a set is open if and only if it contains a ball of some positive radius around each of its points.

We define an *$\epsilon$-perturbation* of the dynamical system at a fixed point $\mathbf{x}^*$ to be a trajectory of the system starting at some $\mathbf{x}_o \in B_\epsilon(\mathbf{x}^*)$, where $\epsilon > 0$ and $\mathbf{x}_o \neq \mathbf{x}^*$. We say a trajectory $\mathbf{x}(t)$ *approaches* $\mathbf{x}^*$ if $\mathbf{x}(t) \to \mathbf{x}^*$ as $t \to \infty$. We say a trajectory $\mathbf{x}(t)$ *$\epsilon$-escapes* $\mathbf{x}^*$ if there is some $t_o$ such that $\mathbf{x}(t) \notin B_\epsilon(\mathbf{x}^*)$ for $t > t_o$; that is, after some point in time, the trajectory never gets closer than $\epsilon$ to $\mathbf{x}^*$.

If there is some $\epsilon > 0$ such that for any $\mathbf{x}_o \in B_\epsilon(\mathbf{x}^*)$, the trajectory through $\mathbf{x}_o$ approaches $\mathbf{x}^*$, we say the fixed point at $\mathbf{x}^*$ is *asymptotically stable*. The set of points $\mathbf{x}_o \in \mathbf{R}^n$ such that a trajectory through $\mathbf{x}_o$ approaches $\mathbf{x}^*$ is called the *basin of attraction* of the fixed point $\mathbf{x}^*$. If every point where the differential equation is defined is in the basin of attraction of $\mathbf{x}^*$, we say the fixed point is *globally stable*.

If $\mathbf{x}^*$ is not asymptotically stable, but for any ball $B_\epsilon(\mathbf{x}^*)$ there is another ball $B_\delta(\mathbf{x}^*)$ such that for any point $\mathbf{x}_o \in B_\delta(\mathbf{x}^*)$, the trajectory starting at $\mathbf{x}_o$

[2] There are many excellent texts on differential equations. Some of my favorites are Perko 1991, Hirsch and Smale 1974, Epstein 1997, and Hofbauer and Sigmund 1998. The last of these is a beautiful summary of evolutionary dynamics.

never leaves $B_\epsilon(\mathbf{x}^*)$, we say the fixed point at $\mathbf{x}^*$ is *neutrally stable*. Neutral stability means that a sufficiently small perturbation about the fixed point never leads the system too far away from the fixed point. A special case is when any trajectory through $\mathbf{x}_o \in B_\epsilon(\mathbf{x}^*)$ is a *closed orbit*; that is, the trajectory starting at $\mathbf{x}_o$ eventually returns to $\mathbf{x}_o$.

If $\mathbf{x}^*$ is neither asymptotically stable nor neutrally stable, we say $\mathbf{x}^*$ is *unstable*. Thus, $\mathbf{x}^*$ is unstable if there is an $\epsilon > 0$ such that for any ball $B_\delta(\mathbf{x}^*)$, there is a point $\mathbf{x}_o \in B_\delta(\mathbf{x}^*)$ such that the trajectory starting at $\mathbf{x}_o$ $\epsilon$-escapes $\mathbf{x}^*$.

## 11.6 Existence and Uniqueness

THEOREM 11.1 Existence, Uniqueness, and Continuous Dependence on Initial Conditions. *Suppose that $f$ in equation (11.1) has continuous derivatives on an open set $D$ containing a point $x_o$. Then there is some interval $I = [-t_o, t_o]$ and a unique trajectory $\mathbf{x}(t)$ satisfying (11.1) defined on $I$ with $\mathbf{x}(0) = x_o$. Moreover, $\mathbf{x}(t)$ depends smoothly upon $x_o$ in the following sense: there is some $\delta > 0$, and a unique function $\mathbf{x}(t, \mathbf{y})$ that satisfies (11.1) on an interval $[-t_1, t_1]$ with $\mathbf{x}(0, \mathbf{y}) = \mathbf{y}$, for all $\mathbf{y} \in B_\delta(\mathbf{x}_o)$. Moreover, $\mathbf{x}(t, \mathbf{y})$ has continuous partial derivatives, and continuous second partial derivatives with respect to $t$.*

This theorem says that if $f(\mathbf{x})$ is suitably well behaved, the dynamical system (11.1) has a unique, twice-differentiable trajectory through each point $\mathbf{x}_o$, and the trajectory varies differentiably as we vary $\mathbf{x}_o$. In particular, two trajectories can never cross.

THEOREM 11.2 Continuous Dependence on Parameters. *Let $\mu \in \mathbf{R}^k$ be a set of $k$ parameters, and suppose $f(\mathbf{x}, \mu)$ has continuous partial derivatives in a neighborhood of $(\mathbf{x}_o, \mu_o) \in \mathbf{R}^{n+k}$. Then there is a $t_1 > 0$, a $\delta > 0$, an $\epsilon > 0$, and a unique function $\mathbf{x}(t, \mathbf{y}, \mu)$ that satisfies*

$$\dot{\mathbf{x}} = f(\mathbf{x}(t, \mathbf{y}, \mu), \mu) \tag{11.9}$$

*with $\mathbf{x}(0, \mathbf{y}, \mu) = \mathbf{y}$, for $t \in [-t_1, t_1]$, $\mathbf{y} \in B_\delta(\mathbf{x}_o)$, and $\mu \in B_\epsilon(\mu_o)$. Moreover, $\mathbf{x}(t, \mathbf{y}, \mu)$ has continuous partial derivatives.*

This theorem says that if $f(\mathbf{x}, \mu)$ is suitably well behaved, the trajectories of the dynamical system (11.9) vary differentiably as we vary the parameters $\mu$.