Econometrica, Vol. 29, No. 4 (October 1961)

THE STABILITY OF DYNAMIC PROCESSES[1]

BY HIROFUMI UZAWA[2]

In this paper the stability problem of general dynamic processes is investigated by the classical Lyapunov second method. The stability theorem will be applied to handle the price adjustment process in a competitive economy and generalize some of the results recently obtained by K. J. Arrow, H. K. Block, and L. Hurwicz.

1. INTRODUCTION

IN ECONOMIC analysis we are often concerned with the problem of whether or not the values of prices or quantities changing subject to certain economic laws approach equilibrium values.[3] In recent papers by Arrow and Hurwicz [**2**] and Arrow, Block, and Hurwicz [**1**], the stability problem was extensively investigated for the price adjustment processes in the competitive market. Following the tradition of Samuelson [**8**] and Lange [**5**], they formulate the price adjustment processes by systems of differential equations and ask under what economic conditions the stability of the processes does hold. In the theory of optimum resource allocation, dynamic processes which represent the competitive mechanism are also formulated as systems of differential equations and it is questioned whether or not the optimum activity levels are stable (see, e.g., Arrow, Hurwicz, and Uzawa [**4**, Part II], and Arrow and Hurwicz [**3**]).

It may be of some interest to investigate the stability of dynamic processes from a unified point of view. In the present paper we consider a system of differential equations that represents a general dynamic process and gives a sufficient condition for the stability of the system. The condition modifies the Lyapunov second method[4] in such a way that it may be suitably applied to various dynamic processes in economic analysis.

In later sections we consider price adjustment processes in competitive markets and show that the present stability theorem may be applied to handle those cases such as the gross-substitute case, the two-commodity case, and the one-consumer case.

[1] This work was supported by the Office of Naval Research under Task NR-047-004. Part of the results were read at the Chicago meeting of the Econometric Society on December 28, 1958.

[2] The author is indebted to K. J. Arrow, G. Debreu, and L. Hurwicz for their valuable comments and criticisms. He also wishes to acknowledge the referee of this journal for pointing out several mistakes contained in an earlier version of the paper.

[3] The first rigorous treatment of the stability problem in economic analysis was done by Samuelson [**8**].

[4] See Lyapunov [**6**] and Malkin [**7**].

2. STABILITY AND QUASI-STABILITY

Let $f(x) = [f_1(x),\ldots,f_n(x)]$ be an n-vector valued continuous function defined on a set Ω of n-vectors, $x = (x_1,\ldots, x_n)$. Let us now consider a dynamic process represented by the following system of differential equations:

$$\dot{x}_i = f_i(x) \qquad (i = 1,\ldots, n)\,, \tag{1}$$

with an initial position $x^0 = (x_1^0,\ldots, x_n^0)$ in Ω. In most of the applications in economic analysis, Ω is either the set of all nonnegative n-vectors or the set of all positive n-vectors.[5] It will be assumed that:

(A) *For any initial position x^0 in Ω, the process* (1) *has a solution $x(t;x^0)$ for all $t \geqslant 0$, which is uniquely determined by x^0 and is continuous with respect to x^0.*

In what follows, a solution $x(t;x^0)$ to a system of differential equations is always supposed to be defined for all $t \geqslant 0$.

An n-vector $\bar{x} = (\bar{x}_1,\ldots, \bar{x}_n)$ in Ω is called an *equilibrium* if

$$f(\bar{x}) = 0\,. \tag{2}$$

The set of all equilibria is denoted by E. Since $f(x)$ is continuous, the set E is closed in Ω.

The process (1) is called (globally) *stable* if, for any initial position x^0 in Ω, the solution $x(t;x^0)$ to the process (1) converges to an equibrium $\bar{x}$.

The process (1) is in the present paper called *quasi-stable* if, for any initial position x^0 in Ω,

(3) the solution $x(t;x^0)$ is bounded; i.e., there exists a number K such that

$$-K \leqslant x(t;x^0) \leqslant K \qquad \text{for all} \quad t \geqslant 0\,,$$

and

(4) every limit-point of $x(t;x^0)$, as t tends to infinity, is an equilibrium; i.e., if for some sequence $\{t_\nu\colon \nu = 1,2,\ldots\}$ such that $t_\nu \to \infty$, $\lim_{\nu\to\infty} x(t_\nu)$ exists, then $\lim_{\nu\to\infty} x(t_\nu)$ is an equilibrium.

Let us define the distance of x to the set E by[6]

$$V(x) = \inf_{\bar{x}\varepsilon E} |x-\bar{x}|^2\,, \qquad x \,\varepsilon\, \Omega\,.$$

If E is a closed set, we may write

$$V(x) = \min_{\bar{x}\varepsilon E} |x-\bar{x}|^2\,, \qquad x \,\varepsilon\, \Omega\,.$$

[5] An n-vector x is called nonnegative (positive) if all the components are nonnegative (positive).

[6] For any vector $u = (u_1,\ldots, u_n)$, we denote the length of u by $|u|$:

$$|u| = \left(\sum_{i=1}^n u_i^2\right)^{1/2}.$$

If the set Ω is closed, or generally if the solution to the process (1) remains in a closed set in Ω, then the condition (3) for quasi-stability may be replaced by

$$\lim_{t\to\infty} V[x(t;x^0)] = 0 . \tag{5}$$

In fact, let the condition (5) be satisfied. Then we have by the continuity of $V(x)$ that for any limit-point x^* of $x(t;x^0)$

$$V(x^*) = \lim_{\nu\to\infty} V[x(t_\nu;x^0)] = \lim_{t\to\infty} V[x(t;x^0)] = 0 .$$

Hence

$$x^* \varepsilon E .$$

On the other hand, let the condition (4) be satisfied. Take any limit-value V^* of $V[x(t;x^0)]$ and consider a sequence $\{t_\nu; \nu = 1,2,\ldots\}$ such that $t_\nu \to \infty$ $(\nu \to \infty)$ and $V^* = \lim_{\nu\to\infty} V[x(t_\nu;x^0)]$. Since, by (3), $x(t;x^0)$ is bounded, there exists a subsequence $\{t_{\nu_k}; k = 1,2,\ldots\}$ such that $\lim_{k\to\infty} x(t_{\nu_k};x^0)$ exists and is equal to, say x^*:

$$x^* = \lim_{k\to\infty} x(t_{\nu_k};x^0) .$$

By the condition (4) we have

$$x^* \varepsilon E .$$

Therefore, by the continuity of $V(x)$,

$$V^* = \lim_{\nu\to\infty} V[x(t_\nu;x^0)] = \lim_{k\to\infty} V[x(t_{\nu_k};x^0)] = V(x^*) = 0 ,$$

which shows that the relation (5) holds.

If the process is stable, it is quasi-stable. On the other hand, if the set E of equilibria is finite, or more generally countable, then the quasi-stability is equivalent to stability.[7]

3. A STABILITY THEOREM

In this section we shall be concerned with a modification of the Lyapunov second method.

STABILITY THEOREM 1: *Let $f(x)$ satisfy conditions* (A) *and:*

(B) *For any initial position x^0 in Ω, the solution $x(t;x^0)$ to the system* (1) *is contained in a compact set in Ω:*[8] and

[7] This remark is due to K. J. Arrow.

[8] A set of vectors is called *compact* if it is closed and bounded.

(C) *There exists a continuous function $\Phi(x)$ defined on Ω such that, for any $x^0 \varepsilon \Omega$, the function*

$$\varphi(t) = \Phi[x(t;x^0)]$$

is a strictly decreasing function with respect to t unless $x(t;x^0)$ is an equilibrium.
Then the process (1) *is quasi-stable.*

Proof. Let $x(t) = x(t;x^0)$ be the solution to the process (1) with an initial position x^0 in Ω, and $\varphi(t) = \Phi[x(t)]$.

By condition (B) the set $\{x(t) : 0 \leqslant t < \infty\}$ is contained in a compact set in Ω. The continuity of the function $\Phi(x)$, therefore, implies that the set $\{\Phi[x(t)]\}$ is also bounded. On the other hand, by condition (C), $\varphi(t)$ is a non-increasing function of t. Hence $\lim_{t\to\infty} \varphi(t)$ exists, and is equal to, say, φ^*:

$$\varphi^* = \lim_{t\to\infty} \varphi(t) .$$

We shall show that any limit point x^* of $x(t)$ is an equilibrium. Let x^* be a limit point of $x(t)$ as t tends to infinity; i.e., there exists a sequence $\{t_\nu : \nu = 1,2,\ldots\}$, $t_\nu \to \infty$ $(\nu \to \infty)$, such that

$$x^* = \lim_{\nu\to\infty} x(t_\nu) .$$

By (B), x^* is contained in Ω. Consider the solution $x^*(t) = x(t;x^*)$ to the process (1) with initial position x^*, and let

$$\varphi^*(t) = \Phi[x^*(t)] .$$

Since $\Phi(x)$ and $x(t;x^0)$ are continuous with respect to x and x^0, respectively, we have

$$\varphi^*(t) = \Phi[x(t;x^*)] = \Phi[\lim_{\nu\to\infty} x(t;x(t_\nu))] = \lim_{\nu\to\infty} \Phi[x(t;x(t_\nu))] .$$

But, since the solution is uniquely determined by the initial position, we have

$$x(t;x(t_\nu)) = x(t+t_\nu;x^0) .$$

Hence

$$\varphi^*(t) = \lim_{\nu\to\infty} \Phi[x(t+t_\nu;x^0)] = \lim_{\nu\to\infty} \varphi(t+t_\nu) = \varphi^*, \qquad \text{for all } t \geqslant 0.$$

Therefore, by condition (C), $x^* = x^*(0)$ is an equilibrium. This proves that the process (1) is quasi-stable. *Q.E.D.*

The function $\Phi(x)$ satisfying condition (C) will be referred to as a *modified Lyapunov function* with respect to the process (1). It may be noted that, when $\Phi(x)$ is differentiable with respect to $x \varepsilon \Omega$, condition(C) is implied by the following

(C′) $$\sum_{i=1}^{n} \frac{\partial \Phi}{\partial x_i} f_i(x) < 0 , \qquad \text{for all non-equilibrium } x \varepsilon \Omega .$$

The Lyapunov second method requires the existence of a function $\Phi(x)$ which *is definite, differentiable, and satisfies condition* (C'). He confines himself to the cases where the equilibrium is uniquely determined; and his concept of stability is different from that of asymptotic stability.

4. QUASI-INTEGRABLE PROCESSES

As an application of the Stability Theorem 1 proved in the previous section, we consider a certain class of dynamic processes.

The process (1) is called *quasi-integrable* if there exists a *differentiable* function $\Phi(x)$ and n positive functions $\lambda_1(x),\ldots,\lambda_n(x)$, all defined on Ω, such that

(6) $$\frac{\partial \Phi}{\partial x_i} = -\lambda_i(x) f_i(x)\,, \quad i = 1,2,\ldots,n, \text{ for all } x \in \Omega\,.$$

The function $\Phi(x)$ satisfying (6) is a modified Lyapunov function with respect to the process (1). In fact, we have

$$\sum_{i=1}^{n} \frac{\partial \Phi}{\partial x_i} f_i(x) = -\sum_{i=1}^{n} \lambda_i(x) f_i^2(x) < 0$$

unless x is an equilibrium.

Therefore, by applying the Stability Theorem, it can be shown that if the process (1) is quasi-integrable and conditions (A) and (B) are satisfied, then the process (1) is quasi-stable.

In particular, let us consider the case in which Ω is the set of all positive n-vectors and $f(x) = (f_1(x),\ldots,f_n(x))$ is differentiable and has a symmetrical matrix of partial derivatives:

(7) $$\frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i}\,, \quad \text{for all } i,j = 1,\ldots,n, \text{ and } x \in \Omega\,.$$

Then it is well known that there exists a continuously differentiable function $\Phi(x)$ on Ω such that

(8) $$\frac{\partial \Phi}{\partial x_i} = -f_i(x)\,, \quad \text{for all } i = 1,\ldots,n, \text{ and } x \in \Omega\,.$$

Hence, in this case, the process (1) is quasi-integrable.

5. THE STABILITY OF PRICE ADJUSTMENT PROCESSES: THE NON-NORMALIZED CASE

In the next two sections we shall investigate the stability of the price adjustment process in a competitive economy.

Let us consider an economic system in which commodities are well defined and for any price system the excess amount of demand over supply is uniquely determined. Commodities are denoted by $i = 0,1,\ldots, n$; a price system is denoted by an $n+1$-vector $p = (p_0, p_1,\ldots, p_n)$ and $h_i(p)$ denotes the excess amount of demand over supply for commodity i when the prices system p prevails in the economy, $i = 0,1,\ldots, n$. We assume that *the excess demand function* $h(p) = (h_0(p),h_1(p),\ldots, h_n(p))$ *is defined at all positive price vectors* $p = (p_0,p_1,\ldots, p_n) > 0$. It will be further assumed that

(H) $h_i(p)$ *is positively homogeneous of order* 0: $h_i(\lambda p) = h_i(p)$, *for any price vector* $p > 0$ *and any positive number* $\lambda > 0$, $i = 0,1,\ldots, n$.

(E) *There exists a positive equilibrium price vector* $\bar{p} = (\bar{p}_0,\bar{p}_1,\ldots, \bar{p}_n) > 0$.

We consider the price adjustment process in which the price of a commodity rises when there is a positive excess demand, and falls when there is a negative excess demand (a positive excess supply). As Samuelson [**8**], Lange [**5**], and Arrow and Hurwicz [**2**] suggest, we formulate the competitive price adjustment process by a system of differential equations.

We first consider a non-normalized price adjustment process defined by the following system of differential equations:

$$\frac{dp_i}{dt} = f_i(p) \qquad (i = 0,1,\ldots,n), \tag{9}$$

where $f_i(p)$ is a function defined at any positive price vector p and has the same sign as the excess demand function $h_i(p)$:

$$\operatorname{sign} f_i(p) = \operatorname{sign} h_i(p) \qquad (i = 0,1,\ldots,n).$$

It will be assumed again that condition (A) is satisfied for the process (7) and $\Omega = \{p = (p_0,p_1,\ldots, p_n) : p > 0\}$.

The Strongly Gross Substitute Case.

It is said that all commodities are *strongly gross substitutes if the excess demand function* $h(p) = (h_0(p),h_1(p),\ldots,h_n(p))$ *is differentiable at all positive price vectors* $p = (p_0,p_1,\ldots, p_n) > 0$; *and*

$$\frac{\partial h_i}{\partial p_j} > 0, \qquad \text{for } i \neq j \text{ and } p > 0.$$

In the strongly gross substitute case we have the following:

LEMMA 1: *If all commodities are strongly gross substitutes and conditions* (H) *and* (E) *are satisfied, then the equilibrium price vector* $\bar{p} = (\bar{p}_0,\bar{p}_1,\ldots, \bar{p}_n)$ *is uniquely* (*up to a scalar multiple*) *determined, and, for any positive price vector* $p = (p_0,p_1,\ldots, p_n)$ *which is not an equilibrium price vector, we have*

$$h_i(p) < 0, \ h_k(p) > 0,$$

where i and k are indices such that

(10) $$\frac{p_i}{\bar{p}_i} = \max_{j=0,1,\cdots,n} \frac{p_j}{\bar{p}_j}, \frac{p_k}{\bar{p}_k} \min_{j=0,1,\cdots,n} \frac{p_j}{\bar{p}_j}.$$

Lemma 1 has been proved by Arrow, Block, and Hurwicz [**1**], and will be used in proving the following stability theorem:[9]

STABILITY THEOREM 2: *If all commodities are strongly gross substitutes and conditions* (A), (H), *and* (E) *are satisfied, then the price adjustment process* (9) *is stable.*

Proof. Let $\bar{\Lambda}(p)$ and $\underline{\Lambda}(p)$ be two functions defined by

(11) $$\bar{\Lambda}(p) = \max_{j=0,1,\cdots,n} \frac{p_j}{\bar{p}_j},$$

and

(12) $$\underline{\Lambda}(p) = \min_{j=0,1,\cdots,n} \frac{p_j}{\bar{p}_j},$$

where $\bar{p}$ is a positive equilibrium price vector. We shall show $\bar{\Lambda}(p)$ and $-\underline{\Lambda}(p)$ are modified Lyapunov functions with respect to the process (9).

Let $p(t) = p[p;p^0]$ be the solution to the process (9) with an arbitrary positive initial vector p^0 and $\bar{\lambda}(t) = \bar{\Lambda}[p(t;p^0)]$, $\underline{\lambda}(t) = \underline{\Lambda}[p(t;p^0)]$. It will suffice to show that

(13) $$\overline{\lim_{h\to 0}} \frac{\bar{\lambda}(t+h) - \bar{\lambda}(t)}{h} \leqslant 0,$$

and

(14) $$\underline{\lim_{h\to 0}} \frac{(\underline{\lambda}(t+h) - \underline{\lambda}(t)}{h} \geqslant 0,$$

with strict inequalities if $p(t)$ is not an equilibrium.

Since (13) and (14) are proved similarly, we shall, for example, prove the inequality (13). From (11) for any t there exists an index i such that[10]

$$\overline{\lim_{h\to 0}} \frac{\bar{\lambda}(t+h) - \bar{\lambda}(t)}{h} = \frac{1}{p_i} \frac{dp_i}{dt},$$

where

$$i \,\varepsilon\, I(p(t)) = \left\{ i : \frac{p_i(t)}{p_i} = \max_{j=1,\cdots,n} \frac{p_j(t)}{p_j} \right\}.$$

[9] See Theorem 1 in Arrow, Block, and Hurwicz [**1**, p. 95].

[10] The existence of such an index i was discussed in a similar context by Arrow, Block, and Hurwicz [**1**, pp. 96–97].

In fact, there is a sequence $\{h_\nu\}$ such that

$$h_\nu \to 0 \ (\nu \to \infty) ,$$

and

$$\overline{\lim_{h \to 0}} \frac{\bar{\lambda}(t+h) - \bar{\lambda}(t)}{h} = \lim_{\nu \to \infty} \frac{\bar{\lambda}(t+h_\nu) - \bar{\lambda}(t)}{h_\nu} .$$

Since the number of indices in $I(p(t))$ is finite, there is at least one index i such that $i \varepsilon I(p(t))$ and

$$\lim_{\nu \to \infty} \frac{\bar{\lambda}(t+h_\nu) - \bar{\lambda}(t)}{h_\nu} = \lim_{\nu \to \infty} \frac{1}{\bar{p}_i}\left(\frac{p_i(t+h_\nu) - P_i(t)}{h_\nu}\right) = \lim_{h \to 0} \frac{1}{\bar{p}_i} \frac{p_i(t+h) - p_i(t)}{h}$$

(by the differentiability of $p_i(t)$)

$$= \frac{1}{\bar{p}_i} \frac{dp_i}{dt} .$$

Hence, by (9),

$$\overline{\lim_{h \to 0}} \frac{\bar{\lambda}(t+h) - \bar{\lambda}(t)}{h} = \frac{1}{\bar{p}_i} f_i[p(t)] . \tag{15}$$

Since $f_i[p(t)]$ and $h_i[p(t)]$ have the same sign, we have, by Lemma 1,

$$f_i[p(t)] \leqslant 0 , \tag{16}$$

with strict inequality if $p(t)$ is not an equilibrium. By (15) and (16) we have (13).

The inequalities (13) and (14) now imply that $\bar{\lambda}(t)$ and $(-\underline{\lambda}t)$ are *strictly decreasing* whenever $p(t)$ is not an equilibrium. Hence $\bar{\Lambda}(p)$ and $-\underline{\Lambda}(p)$ are modified Lyapunov functions. We have, in particular,

$$0 < \underline{\lambda}(0) \leqslant (\underline{\lambda}t) \leqslant \bar{\lambda}(t) \leqslant \bar{\lambda}(0) . \tag{17}$$

The solution $p(t)$ is, therefore, contained in a compact set $\{p ; \underline{\Lambda}(p^0) \leqslant p_i/\bar{p}_i \leqslant \bar{\Lambda}(p^0), i = 0,1,\ldots, n\}$ of positive vectors. By applying the Stability Theorem 2, any limit point of $p(t)$, as t tends to infinity, is an equilibrium; in particular, there exists a sequence $\{t_\nu ; \nu = 1,2,\ldots\}$ such that $t_\nu \to \infty$ $(\nu \to \infty)$ and

$$\lim_{\nu \to \infty} \frac{p_j(t_\nu)}{\bar{p}_j} = 1 \qquad (j = 0,1,\ldots,n) .$$

Hence

$$\lim_{\nu \to \infty} \bar{\lambda}(t_\nu) = \lim_{\nu \to \infty} \underline{\lambda}(t_\nu) = 1 .$$

But

$$(\underline{\lambda}t) \leqslant \frac{p_j(t)}{\bar{p}_j} \leqslant \bar{\lambda}(t) , \qquad \text{for all } t \text{ and } j = 0,1,\ldots,n ,$$

and $(\underline{\lambda}t)$ and $\bar{\lambda}(t)$ are both bounded and monotone. Therefore,

$$\lim_{t \to \infty} \frac{p_j(t)}{\bar{p}_j}$$

always exists and equals 1, for $j = 0,1,\ldots, n$. *Q.E.D.*

The Weakly Gross Substitute Case.

It may be of some interest to investigate the stability of the competitive price adjustment process when all commodities are *weakly gross substitutes*; i.e., when *the excess demand function* $h(p)$ *is differentiable at all positive price vectors* $p > 0$, *and*

$$\frac{\partial h_i}{\partial p_j} \geqslant 0\,, \qquad \text{for } i \neq j \text{ and } p > 0\,.$$

In this case the uniqueness of equilibrium price vectors does not hold, and Lemma 1 may be modified as follows:

LEMMA 2. *If all commodities are weakly gross substitutes and conditions* (H) *and* (E) *are satisfied, then, for any positive price vector* $p = (p_0, p_1, \ldots, p_n)$, *we have*

$$h_i(p) \leqslant 0,\ h_k(p) \geqslant 0\,,$$

where i *and* k *are indices such that* (10) *holds.*

We have the following:

STABILITY THEOREM 3: *If all commodities are weakly gross substitutes and conditions* (A), (H), (E), *and*

(W) *the Walras Law*:

$$p \cdot h(p) = \sum_{j=0}^{n} p_j h_j(p) = 0, \textit{ for any } p > 0\,,$$

then the process (9) *is stable, provided*

$$f_j(p) = F_j[h_j(p)],\ F_j' > 0,\ F_j(0) = 0 \qquad (j = 0,1,\ldots,n)\,.$$

Proof. Let $\bar{\Lambda}(p)$ and $\underline{\Lambda}(p)$ be the functions defined by (11) and (12). Then, by Lemma 2, $\bar{\lambda}(t)$ and $-\underline{\lambda}(t)$ are nonincreasing. Hence the solution $p(t)$ to the process (9) with any positive initial position p^0 is contained in a compact set $\{p; \underline{\Lambda}(p^0) \leqslant p_i/\bar{p}_i \leqslant \bar{\Lambda}(p^0),\ i = 0,1,\ldots,n\}$ of positive vectors.

Consider the function $\Phi(p)$ defined by

$$\Phi(p) = \max_{j=0,1,\ldots,n} \frac{f_j(p)}{p_j}\,. \tag{18}$$

It is noted that

$$\Phi(p) \geqslant 0\,, \tag{19}$$

with strict inequality unless p is an equilibrium. In fact, $\Phi(p) \leqslant 0$ implies that

$$f_j(p) \leqslant 0 \qquad (j = 0,1,\ldots,n)\,.$$

We then have

$$h_j(p) \leqslant 0 \qquad\qquad = 0,1,\dots,n)\ ,$$

which, by the Walras Law (W), implies that

$$h_j(p) = 0 \qquad\qquad (j = 0,1,\dots,n)\ .$$

We show that the function $\Phi(p)$ is a modified Lyapunov function with respect to the process (9). Let

$$\varphi(t) = \Phi[p(t)]\ ,$$

and

$$W(t) = \overline{\lim_{h\to 0}}\ \frac{\varphi(t+h) - \varphi(t)}{h}\ .$$

In order to prove that $\Phi(p)$ is a modified Lyapunov function, it will suffice to show that

$$W(t) \leqslant 0\ , \qquad \text{for all } t \geqslant 0\ ,$$

with strict inequality unless $p(t)$ is an equilibrium.

For any t there exists a commodity i such that

$$\frac{f_i(t)}{p_i(t)} = \max_{j=0,1,\dots,n} \frac{f_j(t)}{p_j(t)}$$

and

$$W(t) = \frac{d}{dt}\left[\frac{f_i(t)}{p_i(t)}\right]\ ,$$

where

$$f_j(t) = f_j[p(t)]\ , \qquad h_j(t) = h_j[p(t)]\ .$$

Calculating $(d/dt)\ \{f_i(t)/p_i(t)\}$, we have

$$W(t) = -\frac{f_i(p)}{p_i^2}\dot{p}_i + \frac{1}{p_i}\sum_{j=0}^{n}\frac{\partial f_i}{\partial p_j}\dot{p}_j = -\frac{f_i^2(p)}{p_i^2} + \frac{1}{p_i}\sum_{j=0}^{n}\frac{\partial f_i}{\partial p_j}f_j(p)\ ,$$

where $p = p(t)$.

But,

$$\frac{1}{p_i}\sum_{j=0}^{n}\frac{\partial f_i}{\partial p_j}f_j(p) = \frac{1}{p_i}F_i'\sum_{j=0}^{n}\frac{\partial h_i}{\partial p_j}f_j(p) = \frac{1}{p_i}F_i'\frac{\partial h_i}{\partial p_i}f_i(p) + \sum_{j\neq i}\frac{\partial h_i}{\partial p_j}f_j(p)$$

$$\leqslant \frac{F_i'}{p_i}\frac{\partial h_i}{\partial p_i}f_i(p) + f_i(p)\sum_{j\neq i}\frac{\partial h_i}{\partial p_j}\frac{p_j}{p_i} = F_i'\frac{f_i(p)}{p_i^2}\sum_{j=0}^{n}\frac{\partial h_i}{\partial p_j}p_j = 0\ .$$

Hence,

$$W(t) \leqslant -\left[\frac{p_i(t)}{p_i(t)}\right]^2 = -\varphi(t)^2 < 0\ ,$$

unless $p(t)$ is an equilibrium.

By the Stability Theorem 1 the process (9) is quasi-stable. Let p^* be any limit point of $p(t)$ as t tends to infinity:

$$p^* = \lim_{\nu \to \infty} p(t_\nu) ,$$

for some subsequence $\{t_\nu\}$, such that $t_\nu \to \infty$ $(\nu \to \infty)$.

Define $\bar{\Lambda}^*(p)$ and $\underline{\Lambda}^*(p)$ by

$$\bar{\Lambda}^*(p) = \max_{j=0,1,\dots,n} \frac{p_j}{p_j^*} ,$$

$$\underline{\Lambda}^*(p) = \min_{j=0,1,\dots,n} \frac{p_i}{p_j^*} .$$

Since $\bar{\lambda}^*(t) = \bar{\Lambda}^*[p(t)]$ and $\underline{\lambda}^*(t) = \underline{\Lambda}^*[p(t)]$ are bounded and monotone,

$$\lim_{t \to \infty} \bar{\lambda}^*(t) \qquad \text{and} \qquad \lim_{t \to \infty} \underline{\lambda}^*(t)$$

both exist. But, by the quasi-stability,

$$\lim_{t \to \infty} \bar{\lambda}^*(t) = \lim_{t \to \infty} \underline{\lambda}^*(t) = 1 .$$

Hence,

$$\lim_{t \to \infty} p(t) = p^* . \qquad Q.E.D.$$

The Weak Axiom of Revealed Preference Case.

We shall next investigate the case in which *the weak axiom of revealed preference* for the excess demand holds at an equilibrium:

$$\bar{p} \cdot h(p) = \sum_{j=0}^{n} p_j h_j(p) > 0 , \tag{20}$$

for any equilibrium $\bar{p}$ and non-equilibrium p.

Consider the price adjustment process defined by the following system of differential equations:

$$\frac{dp_i}{dt} = \begin{cases} 0 , & \text{if } p_i = 0, f_i(p) < 0 , \\ f_i(p) , & \text{otherwise,} \end{cases} \tag{21}$$

where

$$f_i(p) = r_i h_i(p) \qquad (i = 0,1,\dots,n)$$

with positive rates r_i of speed of adjustment.

Then we can prove that:[11]

STABILITY THEOREM 4: *If the excess demand function* $h(p)$ *satisfies* (H), (E), (W), *and the weak axiom of revealed preference at an equilibrium, and condition* (A) *is satisfied for the process* (21), *then the process* (21) *is stable.*

[11] See Arrow, Block, and Hurwicz [1, pp. 102–3].

Proof. Let $\bar{p}$ be any equilibrium and $\Phi(p)$ be defined by

$$\Phi(p) = \tfrac{1}{2} \sum_{i=0}^{n} \frac{1}{r_i} (p_i - \bar{p}_i)^2 . \tag{22}$$

Then, denoting summing over those indices i for which $p_i > 0$ or $p_i = 0$ and $f_i(p) > 0$ by Σ_i', we have

$$\begin{aligned} \sum_i{}' \frac{\partial \Phi}{\partial p_i} f_i(p) = \sum_i{}' (p_i - \bar{p}_i) h_i(p) &= \sum_{i=0}^{n} (p_i - \bar{p}_i) h_i(p) + \sum_{p_i=0,\, h_i(p)<0} \bar{p}_i h_i(p) \\ &\leqslant \sum_{i=0}^{n} (p_i - \bar{p}_i) h_i(p) \leqslant - \sum_{i=0}^{n} \bar{p}_i h_i(p) . \end{aligned}$$

By the weak axiom (20), $\Phi(p)$ is a modified Lyapunov function with respect to process (21). The Walras Law (W), however, implies that, for any solution $p(t)$ to the process (21),

$$\sum_{i=0}^{n} \frac{1}{r_i} p_i^2 (t) = \sum_{i=0}^{n} \frac{1}{r_i} p_i^2 (0) ;$$

hence condition (B) is satisfied. By applying the Stability Theorem 1, the process (21) is quasi-stable, i.e., any limit point of $p(t)$ is an equilibrium.

Let p^* be a limit point of $p(t)$, and $\{t_\nu\}$ a subsequence such tha $t_\nu \to \infty$ $(\nu \to \infty)$ and

$$p^* = \lim_{\nu \to \infty} p(t_\nu) .$$

Let the function $\Phi^*(p)$ be defined by (22) with $\bar{p} = p^*$. Then $\Phi^*[p(t)]$ is a nonincreasing function of t; hence $\lim_{t \to \infty} \Phi^*[p(t)]$ exists.

By using the continuity of the solution of the process (21) with respect to the initial position, we have

$$\lim_{t \to \infty} \Phi^*[p(t)] = \Phi^*(p^*) = 0 ,$$

which shows that $p(t)$ itself converges to equilibrium p^*. *Q.E.D.*

The Two-Commodity Case.

Let us consider the stability of the price adjustment process in the case in which the number of commodities is two, and for one of the two commodities, say commodity 1,

(23) $h_1(1,p_1) \lesseqgtr 0$, according to whether p_1 is sufficiently large or small.

Consider the function $\Phi(p_0,p_1)$ defined by

$$\Phi(p_0,p_1) = -\int_0^{p_1/p_0} h_1(1,v)dv, \qquad \text{for } (p_0,p_1) > 0 . \tag{24}$$

Differentiating (24) with respect to p_0 and p_1, respectively, and using the Walras Law (W) and the homogeneity (H), we get

$$\frac{\partial \Phi}{\partial p_0} = \frac{p_1}{p_0^2} h_1\left(1, \frac{p_1}{p_0}\right) = -\frac{1}{p_0} h_0(p_0, p_1) ,$$

$$\frac{\partial \Phi}{\partial p_1} = -\frac{1}{p_0} h_1\left(1, \frac{p_1}{p_0}\right) = -\frac{1}{p_0} h_1(p_0, p_1) .$$

Hence we have

$$(25) \qquad \frac{\partial \Phi}{\partial p_0} f_0(p) + \frac{\partial \Phi}{\partial p_1} f_1(p) = -\frac{1}{p_0} [f_0(p) h_0(p) + f_1(p) h_1(p)] .$$

Since $f_0(p)$ and $f_1(p)$ have the same signs as $h_0(p)$ and $h_1(p)$, respectively, the relation (25) implies that the function $\Phi(p)$ defined by (24) is a modified Lyapunov function. It is also easily shown that the assumption (23) together with (A) implies that the solution to the system (9) remains positive for any positive initial price vector. *The system* (9) *is therefore quasi-stable in the two-commodity case, provided conditions* (A) *and* (23) *are satisfied and the solution is always bounded.*[12]

6. THE STABILITY OF PRICE ADJUSTMENT PROCESSES: THE NORMALIZED CASE

In this section we consider normalized price adjustment processes. Let us take a commodity, say commodity 0, as a *numéraire*, i.e., the price of commodity 0 is to be fixed at one:

$$p_0 = 1 .$$

By a price vector p is now meant an n-vector with positive components:

$$p = (p_1, \ldots, p_n) ; \qquad p_i > 0 \qquad (i = 1, 2, \ldots, n) .$$

The excess demand function $h(p)$ may be considered in this case as an n-vector valued function defined on positive n-vectors $p = (p_1, \ldots, p_n)$:

$$h(p) = (h_1(p), \ldots, h_n(p)) .$$

The competitive price adjustment process may be represented by the following system of differential equations:

$$(26) \qquad \frac{dp_i}{dt} = f_i(p) \qquad (1 = 1, \ldots, n) ,$$

with positive initial price vector $p^0 = (\overset{0}{p_1}, \ldots, \overset{0}{p_n})$, where $f_i(p)$ are functions defined at any positive price vectors and

$$\operatorname{sign} f_i(p) = \operatorname{sign} h_i(p) \qquad (i = 1, \ldots, n) .$$

[12] See *Remark* on [**1**, p. 108]. The boundedness assumption, however, seems to be needed. For this point, the author particularly owes gratitude to the referee.

It will again be assumed that condition (A) is satisfied for the process (26).

By using the Stability Theorem 1 we can investigate the stability of the process (26) in those cases which we investigated in Section 5. For example, let us consider the strongly gross substitute case.

STABILITY THEOREM 5: *If all commodities are strongly gross substitute and conditions* (A), (H), *and* (E) *are satisfied, then the process* (26) *is stable.*

Proof. Let the functions $\bar{\Lambda}(p)$ and $(\underline{\Lambda}p)$ be defined by (11) and (12), where

$$p_0(t) = \bar{p}_0 = 1\ .$$

It is easily derived from Lemma 1 that

$$\overline{\lim_{h\to 0}} \frac{\bar{\lambda}(t+h) - \bar{\lambda}(t)}{h} \leqslant 0 \tag{27}$$

with strict inequality if $\bar{\lambda}(t) > 1$ and

$$\underline{\lim_{h\to 0}} \frac{\underline{\lambda}(t+h) - \underline{\lambda}(t)}{h} \geqslant 0\ , \tag{28}$$

with strict inequality if $\underline{\lambda}(t) < 1$. Hence the solution $p(t)$ is contained in a compact set of positive vectors.

The relations (27) and (28) show that the function $\Phi(p)$ defined by

$$\Phi(p) = \bar{\Lambda}(p) - \underline{\Lambda}(p)$$

is a modified Lyapunov function. The Stability Theorem 1 may, therefore, be applied and we know that the process (26) is quasi-stable. Since the normalized prince vector $\bar{p} = (\bar{p}_1, \ldots, \bar{p}_n)$ is uniquely determined, the process (26) is stable. *Q.E.D.*

Stanford University

REFERENCES

[1] ARROW, K. J., H. D. BLOCK, AND L. HURWICZ: "On the Stability of Competitive Equilibrium, II," *Econometrica*, 27 (1959), 82–109.

[2] ARROW, K. J., AND L. HURWICZ: "On the Stability of Competitive Equilibrium, I," *Econometrica*, 26 (1958), 522–552.

[3] ARROW, K. J., AND L. HURWICZ: "Decentralization and Computation in Resource Allocation," in R. W. Pfouts (ed.) *Essays in Economics and Econometrics: A Volume in Honor of Harold Hotelling*. Chapel Hill: University of North Carolina Press, 1961, pp. 34–104.

[4] ARROW, K. J., L. HURWICZ, AND H. UZAWA: *Studies in Linear and Non-Linear Programming*. Stanford: Stanford University Press, 1958.

[5] LANGE, O.: *Price Flexibility and Employment*. Bloomington: The Principia Press, 1944.

[6] LYAPUNOV, A.: "Problème général de la stabilité du mouvement," *Annales de la Faculté des Sciences de l'Université de Toulouse* (2), 9 (1907), 27–474.
[7] MALKIN, I.: "On the Stability of Motion in the Sense of Lyapunov," (in Russian) *Recueil Mathématique* [*Mat. Sbornik*], *N.S.* (3), 45 (1938), 47–100. Translated as *American Mathematical Society Translation* No. 41, 1951.
[8] SAMUELSON, P. A.: "The Stability of Equilibrium: Comparative Statics and Dynamics," *Econometrica*, 9 (1941), 97–120.