A pure strategy Nash equilibrium of a game is a Nash equilibrium in which each player uses a pure strategy, but not necessarily one determined by the iterated elimination of dominated strategies (§4.1). Not every game has a pure-strategy Nash equilibrium. Indeed, there are even very simple $2 \times 2$ normal form games with no pure-strategy Nash equilibria—for instance throwing fingers (§3.8), where the Nash equilibrium consists in each player throwing one or two fingers, each with probability $1/2$.

This chapter explores some of the more interesting applications of games with pure-strategy equilibria. As you will see, we obtain extremely deep results in various branches of economic theory, including altruism (§5.16, §5.17), the tragedy of the commons (§5.5), the existence of pure-strategy equilibria in games of perfect information (§5.6), the real meaning of competition (it is probably not what you think) (§5.3), honest signaling equilibria (§5.9) and (§5.18). Another feature of this chapter is its use of agent-based modeling, in no-draw, high-low poker (§5.7), to give you a feel for the dynamic properties of games for which the Nash equilibrium concept is a plausible description of reality.

### 5.1 Price Matching as Tacit Collusion

Bernie and Manny both sell DVD players and both have unit costs of 250. They compete on price: the low-price seller gets all the market and they split the market if they have equal prices. Explain why the only Nash equilibrium has both firms charging 250, splitting the market and making zero profit.

Suppose that the monopoly price for DVD players (the price that maximizes the sum of the profits of both firms) is 300. Now suppose Bernie advertises that if a customer buys a DVD player from him for 300 and discovers he or she can buy it cheaper at Manny’s, Bernie will refund the full
purchase price. Suppose Manny does the same thing. Show that it is now Nash for both stores to charge 300. Conclusion: pricing strategies that seem to be supercompetitive can in fact be anticompetitive!

5.2 Competition on Main Street

The residents of Pleasantville live on Main Street, which is the only road in town. Two residents decide to set up general stores. Each can locate at any point between the beginning of Main Street, which we will label 0, and the end, which we will label 1. The two decide independently where to locate and they must remain there forever (both can occupy the same location). Each store will attract the customers who are closest to it and the stores will share equally customers who are equidistant between the two. Thus, for instance, if one store locates at point \( x \) and the second at point \( y > x \), then the first will get a share \( x + (y - x)/2 \) and the second will get a share \( (1 - y) + (y - x)/2 \) of the customers each day (draw a picture to help you see why). Each customer contributes $1.00 in profits each day to the general store it visits.

a. Define the actions, strategies, and daily payoffs to this game. Show that the unique pure-strategy Nash equilibrium where both players locate at the midpoint of Main Street;

b. Suppose there are three General Stores, each independently choosing a location point along the road (if they all choose the same point, two of them share a building). Show that there is no pure-strategy Nash equilibrium. Hint: First show that there is no pure-strategy Nash equilibrium where all three stores locate on one half of Main Street. Suppose two stores locate on the left half of Main Street. Then, the third store should locate a little bit to the right of the rightmost of the other two stores. But, then the other two stores are not best responses. Therefore, the assumption is false. Now finish the proof.

5.3 Markets as Disciplining Devices: Allied Widgets

In *The Communist Manifesto* of 1848, Karl Marx offered a critique of the nascent capitalist order that was to resound around the world and fire the imagination of socialists for nearly a century and a half.
The bourgeoisie, wherever it has got the upper hand, has put an end to all feudal, patriarchal, idyllic relations. It has pitilessly torn asunder the motley feudal ties that bound man to his "natural superiors", and has left no other nexus between man and man than naked self-interest, than callous "cash payment".

...It has resolved personal worth into exchange value, and in place of the numberless indefeasible chartered freedoms, has set up that single, unconscionable freedom: Free Trade. (Marx 1948)

Marx’s indictment covered the two major institutions of capitalism: market competition and private ownership of businesses. Traditional economic theory held that the role of competition was to set prices, so supply equals demand. If this were correct, a socialist society that took over ownership of the businesses could replace competition by a central-planning board that sets prices using statistical techniques to assess supply and demand curves.

The problem with this defense of socialism is that traditional economic theory is wrong. The function of competition is to reveal private information concerning the shape of production functions and the effort of the firm’s managers and use that information to reward hard work and the efficient use of resources by the firm. Friedrich von Hayek (1945) recognized this error and placed informational issues at the heart of his theory of capitalist competition. By contrast, Joseph Schumpeter (1942), always the bitter opponent of socialism, stuck to the traditional theory and predicted the inevitable victory of the system he so hated (Gintis 1991).

This problem pins down analytically the notion that competition is valuable because it reveals otherwise private information. In effect, under the proper circumstances, market competition subjects firms to a prisoner’s dilemma in which it is in the interest of each producer to supply high effort, even in cases where consumers and the planner cannot observe or contract for effort itself. This is the meaning of Bengt Holmström’s quotation at the head of this chapter.

If Holmström is right, and both game-theoretic modeling and practical experience indicate that he is, the defense of competitive markets in neoclassical economics is a great intellectual irony. Because of Adam Smith, supporters of the market system have defended markets on the grounds that they allocate goods and services efficiently. However, empirical estimates of the losses from monopoly, tariffs, quotas, and the like indicate that misal-
Chapter 5

location has little effect on per capita income or the rate of economic growth (Hines 1999). By contrast, the real benefits of competition, which include its ability to turn private into public information, have come to light only through game-theoretic analysis. The following problem is a fine example of such analysis.

Allied Widgets has two possible constant returns to scale production techniques: fission and fusion. For each technique, Nature decides in each period whether marginal cost is 1 or 2. With probability $\theta \in (0, 1)$, marginal cost is 1. Thus, if fission is high cost in a given production period, the manager can use fusion, which will be low cost with probability $\theta$. However, it is costly for the manager to inspect the state of Nature and if he fails to inspect, he will miss the opportunity to try fusion if the cost of fission is high.

Allied’s owner cannot tell whether the manager inspected or not, but he does know the resulting marginal cost and can use this to give an incentive wage to the manager. Figure 5.1 shows the manager’s decision tree, which assumes the manager is paid a wage $w_1$ when marginal costs are low and $w_2$ when marginal costs are high, the cost of inspecting is $\alpha$ and the manager has a logarithmic utility function over income: $u(w) = \ln w$.\footnote{The logarithmic utility function is a reasonable choice, because it implies constant relative risk aversion; that is, the fraction of wealth an agent desires to put in a particular risky security is independent of wealth.}

To induce the manager to inspect the fission process, the owner decides to pay the manager a wage $w_1$ if marginal cost is low and $w_2 < w_1$ if
marginal cost is high. But how should the owner choose \( w_1 \) and \( w_2 \) to maximize profits? Suppose the manager’s payoff is \( \ln w \) if he does not inspect, \( \ln (w - \alpha) \) if he inspects and \( \ln w_o \) if he does not take the job at all. In this case, \( w_o \) is called the manager’s reservation wage or fallback position.

The expression that must be satisfied for a wage pair \((w_1, w_2)\) to induce the manager to inspect the fission process is called the incentive compatibility constraint. To find this expression, note that the probability of using a low-cost technique if the manager does not inspect is \( \theta \), so the payoff to the manager from not inspecting (by the expected utility principle) is

\[
\theta \ln w_1 + (1 - \theta) \ln w_2.
\]

If the manager inspects, both techniques will turn out to be high cost with probability \( (1 - \theta)^2 \), so the probability that at least one of the techniques is low cost is \( 1 - (1 - \theta)^2 \). Thus, the payoff to the manager from inspecting (again by the expected utility principle) is

\[
[1 - (1 - \theta)^2] \ln w_1 + (1 - \theta)^2 \ln w_2 - \alpha.
\]

The incentive compatibility constraint is then

\[
\theta \ln w_1 + (1 - \theta) \ln w_2 \leq [1 - (1 - \theta)^2] \ln w_1 + (1 - \theta)^2 \ln w_2 - \alpha.
\]

Because there is no reason to pay the manager more than absolutely necessary to get him to inspect, we can assume this is an equality,\(^2\) in which case the constraint reduces to \( \theta (1 - \theta) \ln [w_1 / w_2] = \alpha \), or

\[
w_1 = w_2 e^{\alpha / (\theta (1 - \theta))}.
\]

For instance, suppose \( \alpha = 0.4 \) and \( \theta = 0.8 \). Then \( w_1 = 12.18 w_2 \); that is, the manager must be paid more than twelve times as much in the good state as in the bad!

But the owner must also pay the manager enough so that taking the job, compared to taking the fallback \( w_o \) is worthwhile. The expression that must be satisfied for a wage pair \((w_1, w_2)\) to induce the manager to take the job is

\[
\theta \ln w_1 + (1 - \theta) \ln w_2 = \ln w_o.
\]

\(^2\)Actually, this point may not be obvious and is false in the case of repeated principal-agent models. This remark applies also to our assumption that the participation constraint, defined in the text, is satisfied as an equality.
called the participation constraint. In our case, the participation constraint is

\[ [1 - (1 - \theta)^2] \ln w_1 + (1 - \theta)^2 \ln w_2 - \alpha \geq \ln w_o. \]

If we assume that this is an equality and using the incentive compatibility constraint, we find \( w_o = w_2 e^{\alpha/(1 - \theta)} \), so

\[ w_2 = w_o e^{-\alpha/(1 - \theta)}, \quad w_1 = w_o e^\phi. \]

Using the above illustrative numbers and if we assume \( w_o = 1 \), we get

\[ w_2 = 0.14, \quad w_1 = 1.65. \]

The expected cost of the managerial incentives to the owner is

\[ [1 - (1 - \theta)^2] w_1 + (1 - \theta)^2 w_2 = w_o \left[ \theta (2 - \theta) e^\phi + (1 - \theta)^2 e^{\alpha/(1 - \theta)} \right]. \]

Again, using our illustrative numbers, we get expected cost

\[ 0.96(1.65) + 0.04(0.14) = 1.59. \]

So where does competition come in? Suppose Allied has a competitor, Axis Widgets, subject to the same conditions of production. In particular, whatever marginal-cost structure Nature imposes on Allied, Nature also imposes on Axis. Suppose also that the managers in the two firms cannot collude. We can show that Allied’s owner can write a Pareto-efficient contract for the manager using Axis’s marginal cost as a signal, satisfying both the participation and incentive compatibility constraints and thereby increasing profits. They can do this by providing incentives that subject the managers to a prisoner’s dilemma, in which the dominant strategy is to defect, which in this case means to inspect fission in search of a low-cost production process.

To see this, consider the following payment scheme, used by both the Axis and the Allied owners, where \( \phi = 1 - \theta + \theta^2 \), which is the probability that both managers choose equal-cost technologies when one manager inspects and the other does not (or, in other words, one minus the probability that the first choice is high and the second low). Moreover, we specify the parameters \( \beta \) and \( \gamma \) so that \( \gamma < -\alpha (1 - \theta + \theta^2) / \theta (1 - \theta) \) and \( \beta > \alpha (2 - \phi) / (1 - \phi) \). This gives rise to the payoffs to the manager shown in the table, where the example uses \( \alpha = 0.4, \theta = 0.8, \) and \( w_0 = 1 \).
We will show that the manager will always inspect and the owner’s expected wage payment is $w^*$, which merely pays the manager the equivalent of the fallback wage. Here is the normal form for the game between the two managers.

<table>
<thead>
<tr>
<th>Allied Cost</th>
<th>Axis Cost</th>
<th>Allied Wage</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c = 1$</td>
<td>$c = 1$</td>
<td>$w^* = w_o e^\alpha$</td>
<td>$w^* = 1.49$</td>
</tr>
<tr>
<td>$c = 2$</td>
<td>$c = 2$</td>
<td>$w^* = w_o e^\alpha$</td>
<td>$w^* = 1.49$</td>
</tr>
<tr>
<td>$c = 1$</td>
<td>$c = 2$</td>
<td>$w^+ = w_o e^\beta$</td>
<td>$w^+ = 54.60$</td>
</tr>
<tr>
<td>$c = 2$</td>
<td>$c = 1$</td>
<td>$w^- = w_o e^\gamma$</td>
<td>$w^- = 0.02$</td>
</tr>
</tbody>
</table>

Why is this so? The inspect/inspect and shirk/shirk entries are obvious. For the inspect/shirk box, with probability $\phi$ the two managers have the same costs, so they each get $\ln w^*$ and with probability $1 - \phi$ the Allied manager has low costs and the Axis manager has high costs, so the former gets $\phi \ln w^+ - \alpha$ and the latter gets $\phi \ln w^- - \alpha$.

To show that this is a prisoner’s dilemma, we need only show that

$$\ln w^* - \alpha > \phi \ln w^* + (1 - \phi) \ln w^-$$

and

$$\phi \ln w^* + (1 - \phi) \ln w^+ - \alpha > \ln w^*.$$  

The first of these becomes

$$\ln w_o > \phi \ln w_o + \phi \alpha + (1 - \phi) \ln w_o + (1 - \phi) \gamma,$$

or $\gamma < -\phi \alpha / (1 - \phi)$, which is true by assumption. The second becomes

$$\ln w^+ > \frac{\alpha}{1 - \phi} + \ln w^*.$$

or $\beta > \frac{\alpha^2}{1 - \phi}$, which is also true by assumption.

Note that in our numerical example the cost to the owner is $w^* = 1.49$ and the incentives for the managers are given by the normal form matrix.
This example shows that markets may be disciplining devices in the sense that they reduce the cost involved in providing the incentives for agents to act in the interests of their employers or clients, even where enforceable contracts cannot be written. In this case, there can be no enforceable contract for managerial inspecting. Note that in this example, even though managers are risk averse, imposing a structure of competition between the managers means each inspects and the cost of incentives is no greater than if a fully specified and enforceable contract for inspecting could be written.

Of course, if we weaken some of the assumptions, Pareto-optimality will no longer be attainable. For instance, suppose when a technique is low cost for one firm, it is not necessarily low cost for the other, but rather is low cost with probability \( q > \frac{1}{2} \). Then competition between managers has an element of uncertainty and optimal contracts will expose the managers to a positive level of risk, so their expected payoff must be greater than their fallback.

### 5.4 The Tobacco Market

The demand for tobacco is given by

\[
q = 100000(10 - p),
\]

where \( p \) is the price per pound. However, there is a government price support program for tobacco that ensures that the price cannot go under $0.25 per pound. Three tobacco farmers have each harvested 600,000 pounds of tobacco. Each must make an independent decision on how much to ship to the market and how much to discard.

a. Show that there are two Nash equilibria, one in which each farmer ships the whole crop and a second in which each farmer ships 250,000 pounds and discards 350,000 pounds.

b. Are there any other Nash equilibria?
5.5 The Klingons and the Snarks

Two Klingons are eating from a communal cauldron of snarks. There are 1,000 snarks in the cauldron and the Klingons decide individually the rate $r_i$, ($i = 1, 2$) at which they eat per eon. The net utility from eating snarks, which depends on both the amount eaten and the rate of consumption (too slow depletes the Klingon Reservoir, too fast overloads the Klingon Kishkes) is given by

$$u_i = 4q_i + 50r_i - r_i^2,$$

where $q_i$ is the total number of snarks Klingon $i$ eats. Since the two Klingons eventually eat all the snarks, $q_i = 1000r_i / (r_1 + r_2)$.

a. If they could agree on an optimal (and equal) rate of consumption, what would that rate be?

b. When they choose independently, what rate will they choose?

c. This problem illustrates the tragedy of the commons (Hardin 1968), in which a community (in this case the two Klingons, though it usually involves a larger number of individuals) overexploits a resource (in this case the bowl of snarks) because its members cannot control access to the resource. Some economists believe the answer is simple: the problem arises because no one owns the resource. So give an individual the right to control access to the resource and let that individual sell the right to extract resources at a rate $r$ to the users. To see this, suppose the cauldron of snarks is given to a third Master Klingon and suppose the Master Klingon charges a diner a fixed number of drecks (the Klingon monetary unit), chosen to maximize his profits, for the right to consume half the cauldron. Show that this will lead to an optimal rate of consumption.

This “create property rights in the resource” solution is not always satisfactory, however. First, it makes the new owner rich and everyone else poor. This could possibly be solved by obliging the new owner to pay the community for the right to control the resource. Second, it may not be possible to write a contract for the rate of resource use; the community as a whole may be better at controlling resource use than a single owner (Ostrom, Walker, and Gardner 1992). Third, if there is unequal ability to pay among community members, the private property solution may lead to an unequal distribution of resources among community members.
5.6 Chess: The Trivial Pastime

A finite game is a game with a finite number of nodes in its game tree. A game of perfect information is a game where every information set is a single node and Nature has no moves. In 1913 the famous mathematician Ernst Zermelo proved that in chess either the first mover has a winning pure strategy, the second mover has a winning pure strategy, or either player can force a draw. This proof was generalized by Harold Kuhn (1953), who proved that every finite game of perfect information has a pure-strategy Nash equilibrium. In this problem you are asked to prove a special case of this, the game of chess.

Chess is clearly a game of perfect information. It is also a finite game, because one of the rules is that if the board configuration is repeated three times, the game is a draw. Show that in chess, either Black has a winning strategy, or White has a winning strategy, or both players have strategies that can force a draw.

Of course, just because there exists an optimal strategy does not imply that there is a feasible way to find one. There are about $10^{47}$ legal positions in chess, give or take a few orders of magnitude, implying a game-tree complexity that is almost the cube of this number. This is far more than the number of atoms in the universe.

5.7 No-Draw, High-Low Poker

Alice and Bob are playing cards. The deck of cards has only two types of card in equal numbers: high and low. Each player each puts 1 in the pot. Alice is dealt a card (by Nature). After viewing the card, which Bob cannot see, she either raises or stays. Bob can stay or fold if Alice raises and can raise or stay if Alice stays. If Alice raises, she puts an additional 1 in the pot. If Bob responds by folding, he looses and if he responds by staying, he must put an additional 1 in the pot. If Alice stays and Bob raises, both must put an additional 1 in the pot. If the game ends without Bob folding, Alice wins the pot if she has a high card and loses the pot if she has a low card. Each player’s objective is to maximize the expected value of his or her winnings. The game tree is in figure 5.2

We now define strategies for each of the players. Alice has two information sets (each one a node) and two choices at each. This gives four strategies, which we label $RR$, $RS$, $SR$, and $SS$. These mean “raise no
manner what,” “raise with high, stay with low,” “stay with high, raise with low,” and “stay no matter what.” Bob also has two information sets, one where Alice raises and one where Alice stays. We denote his four strategies: $SR$, $SS$, $FR$, and $FS$. These mean “stay if Alice raises, raise if Alice stays,” “stay no matter what,” “fold if Alice raises, raise if Alice stays,” and “fold if Alice raises, stay if Alice stays.” To find the normal form game, we first assume Nature gives Alice a high card and compute the normal form matrix. We then do the same assuming Nature plays low. This gives the following payoffs:

![Game Tree](image)

The expected values of the payoffs for the two players are simply the averages of these two matrices of payoffs, because Nature chooses high or Low each with probability 1/2. We have:

<table>
<thead>
<tr>
<th>Nature Plays High</th>
<th>Nature Plays Low</th>
</tr>
</thead>
<tbody>
<tr>
<td>$SR$</td>
<td>$SR$</td>
</tr>
<tr>
<td>$2, -2$</td>
<td>$-2, 2$</td>
</tr>
<tr>
<td>$2, -2$</td>
<td>$-2, 2$</td>
</tr>
<tr>
<td>$RR$</td>
<td>$RR$</td>
</tr>
<tr>
<td>$2, -2$</td>
<td>$-2, 2$</td>
</tr>
<tr>
<td>$2, -2$</td>
<td>$-2, 2$</td>
</tr>
<tr>
<td>$2, -2$</td>
<td>$-2, 2$</td>
</tr>
<tr>
<td>$1, -1$</td>
<td>$1, -1$</td>
</tr>
<tr>
<td>$SS$</td>
<td>$SS$</td>
</tr>
<tr>
<td>$2, -2$</td>
<td>$-2, 2$</td>
</tr>
<tr>
<td>$2, -2$</td>
<td>$-2, 2$</td>
</tr>
<tr>
<td>$2, -2$</td>
<td>$-2, 2$</td>
</tr>
<tr>
<td>$1, -1$</td>
<td>$1, -1$</td>
</tr>
</tbody>
</table>

The expected values of the payoffs for the two players are simply the averages of these two matrices of payoffs, because Nature chooses high or Low each with probability 1/2. We have:
Some strategies in this game can be dropped because they are recursively dominated. For Alice, $RR$ weakly dominates $SS$ and for Bob, $SR$ weakly dominates $FS$. It is quite straightforward to check that $\{RR, SR\}$ is a Nash equilibrium. Note that the game is fair: Alice raises no matter what and Bob stays if Alice raises and raises if Alice stays. A box-by-box check shows that there is another pure-strategy equilibrium, $\{SS, SR\}$, in which Alice uses a weakly dominated strategy. There are also some mixed strategy equilibria for which you are invited to search.

### 5.8 An Agent-based Model of No-Draw High-Low Poker

The heavy emphasis on finding Nash equilibria in evolutionary game theory flows from two assertions. First, the equilibria of dynamic evolutionary games are always Nash equilibria (§12.8). Second, the evolutionary process does not require high-level rationality from the agents who populate dynamic evolutionary games. We can illustrate both points by modeling the dynamics of no-draw, high-low poker on the computer. In this agent-based model, I created 100 player 1 types and 100 player 2 types, each programmed to play exactly one pure strategy, assigned randomly to them. In each round of play, player 1’s and player 2’s are randomly paired and they play no-draw high-low poker once. Every 100 rounds we allow reproduction to take place. Reproduction consisted in killing off 5% of the players of each type with the lowest scores and allowing the top 5% of players with the highest score to reproduce and take the place of the defunct low scorers. However with a 1% probability, a newly-born player ‘mutates’ by using some randomly-chosen other strategy. The simulation ran for 50,000 rounds. The results of a typical run of the simulations for the distribution of player 2 types in the economy are shown in figure 5.3. Note that the Nash strategy for player 2 slowly but surely wins out and the other strategies remain at very low levels, though they cannot disappear altogether because
mutations constantly occur to replenish their ranks. Although this is not shown in figure 5.3, player 1 uses $RR$ rather than the weakly dominated $SS$.

![No-Draw High-Low Poker](image)

Figure 5.3. An agent-based model of no-draw, high-low poker

### 5.9 The Truth Game

Bob wants a used car and asks the sales representative, Alice, “Is this a good car for the money?” Alice wants to sell the car, but does not want to ruin her reputation by lying. We can model the strategies followed by Alice and Bob as follows.

Nature flips a coin and it comes out $H$ with probability $p = 0.8$ and $T$ with probability $p = 0.2$. Alice sees the result, but Bob does not. Alice announces to Bob that the coin came out either $H$ or $T$. Then Bob announces either $h$ or $t$. The payoffs are as follows: Alice receives 1 for telling the truth and 2 for inducing Bob to choose $h$. Bob receives 1 for making a correct guess and 0 otherwise.

The game tree for the problem is depicted in figure 5.4. Alice has strategy set $\{HH, HT, TH, TT\}$, where $HH$ means announce $H$ if you see $H$, announce $H$ if you see $T$, $HT$ means announce $H$ if you see $H$, announce $T$ if you see $T$, and so on. Thus, $HH$ means “always say $H$,” $HT$ means “tell the truth,”
TH means “always lie;” TT means “always say T.” Bob has strategy set \{hh, ht, th, tt\}, where hh means “say h if you are told h and say h if you are told t;” ht means “say h if you are told h and say t if you are told t;” and so on. Thus, hh means “always say h,” ht means “trust Alice,” th means “distrust Alice,” and tt means “always say t.” The payoffs to the two cases, according to Nature’s choice, are listed in figure 5.5.

\[
\begin{array}{cccc}
\text{HH} & \text{HT} & \text{TH} & \text{TT} \\
2.0 & 3.0 & 2.0 & 3.0 \\
2.0 & 1.1 & 0.1 & 1.1 \\
0.1 & 3.0 & 0.1 & 1.1 \\
0.1 & 1.1 & 0.1 & 1.1 \\
\end{array}
\]

Payoff when coin is T

\[
\begin{array}{cccc}
\text{HH} & \text{HT} & \text{TH} & \text{TT} \\
3.1 & 3.1 & 3.1 & 3.1 \\
3.1 & 3.1 & 3.1 & 3.1 \\
1.0 & 1.0 & 1.0 & 1.0 \\
1.0 & 1.0 & 1.0 & 1.0 \\
\end{array}
\]

Payoff when coin is H

The actual payoff matrix is \(0.2 \times \text{first matrix} + 0.8 \times \text{second},\) which is shown in figure 5.6.

\[
\begin{array}{cccc}
\text{HH} & \text{HT} & \text{TH} & \text{TT} \\
2.8,0.8 & 2.8,0.8 & 0.8,0.2 & 0.8,0.2 \\
3.0,0.8 & 2.6,1.0 & 1.4,0.0 & 1.0,0.2 \\
2.0,0.8 & 0.4,0.0 & 1.6,1.0 & 0.0,0.2 \\
2.2,0.8 & 0.2,0.8 & 2.2,0.8 & 0.2,0.2 \\
\end{array}
\]

Figure 5.6. Consolidated payoffs for the truth game

It is easy to see that tt is dominated by hh, but there are no other strategies dominated by pure strategies. (TT, th) and (HH, ht) are both Nash equilibria. In the former, Alice always says T and Bob assumes Alice lies; in
the second, Alice always says H and Bob always believes Alice. The first equilibrium is Pareto-inferior to the second.

It is instructive to find the effect of changing the probability \( p \) and/or the cost of lying on the nature of the equilibrium. You are encouraged so show that if the cost of lying is sufficiently high, Alice will always tell the truth and in equilibrium, Bob will believe her.

5.10 The Rubinstein Bargaining Model

Suppose Bob and Alice bargain over the division of a dollar (Rubinstein 1982). Bob goes first and offers Alice a share \( x \) of the dollar. If Alice accepts, the payoffs are \( (1 - x, x) \) and the game is over. If Alice rejects, the pie “shrinks” to \( \delta < 1 \) and Alice offers Bob a share \( y \) of this smaller pie. If Bob accepts, the payoffs are \( (y, \delta - y) \). Otherwise, the pie shrinks to \( \delta^2 \) and it is once again Bob’s turn to make an offer. The game continues until they settle, or if they never settle, the payoff is (0,0). The game tree is shown in figure 5.7.

Clearly, for any \( x \in [0, 1] \) there is a Nash equilibrium in which the payoffs are \( (1 - x, x) \), simply because if Alice accepts nothing less than \( x \), then it is Nash for Bob to offer \( x \) to Alice and conversely. But these equilibria are not necessarily credible strategies, because, for instance, it is not credible to demand more than \( \delta \), which is the total size of the pie if Alice rejects the offer. What are the plausible Nash equilibria? In this case, equilibria are implausible because they involve incredible threats, so in this case forward induction on the part of Bob and Alice lead them both to look for subgame perfect equilibria.

For the subgame perfect case, let \( \frac{1 - x}{\frac{1 - \delta}{1 + \delta}} = \frac{1}{1 + \delta} \).

Now let \( 1 - x \) be the least Bob can possibly get in any subgame perfect Nash equilibrium. Then, the least Bob can get on his second turn to offer is \( \delta^2 (1 - x) \), so on Alice’s first turn, the most she must offer Bob is \( \delta^2 (1 - x) \), so the least Alice gets when it is her turn to offer is \( \delta - \delta^2 (1 - x) \). But then, on his first turn, Bob must offer Alice at least this amount, so his payoff is at most \( 1 - \delta + \delta^2 (1 - x) \). But this must equal \( 1 - x \), so we have \( x = \frac{1 - \delta}{1 + \delta} = \frac{1}{1 + \delta} \).
payoff is at least $1 - \delta + \delta^2 (1 - x)$. But this must equal $1 - x$, so we have \[ x = \frac{1-\delta}{1-\delta^2} = \frac{1}{1+\delta}. \]

Because the least Bob can earn and the most Bob can earn in a subgame perfect Nash equilibrium are equal, there is a unique subgame perfect Nash equilibrium, in which the payoffs are

\[
\left( \frac{1}{1+\delta}, \frac{\delta}{1+\delta} \right).
\]

Note that there is a small first-mover advantage, which disappears as $\delta \to 1$.

Do people actually find this subgame perfect equilibrium? Because it requires only two levels of backward induction, we might expect people to do so, despite that fact that it involves the iterated elimination of weakly dominated strategies (note that if the players are Bayesian rational, we never get to Bob’s second move). But in fact, the game tree is infinite and our trick to reducing backward induction to two steps is purely formal. Thus, we might not expect people to settle on the Rubinstein solution. Indeed, experimental evidence indicates that they do not (Neelin, Sonnenschein, and Spiegel 1988; Babcock, Loewenstein, and Wang 1995). Part of the reason is that fairness issues enter into many bargaining situations. These issues are usually not important in the context of the current game, because unless...
the discount factor is very low, the outcome is almost a fifty-fifty split. But if we complicated the model a bit—for instance, by giving players unequal “outside options” that occur with positive probability after each rejected offer—very unequal outcomes become possible. Also, the basic Rubinstein model predicts that all bargaining will be efficient, because the first offer is in fact never refused. In real-world bargaining, however, breakdowns often occur (strike, war, divorce). Generally, we need models of bargaining with asymmetric information or outside options to have breakdowns with positive probability (see section 5.13 and section 6.42).

5.11 Bargaining with Heterogeneous Impatience

Suppose in the Rubinstein bargaining model we assume Bob has discount factor \( \delta_B \) and Alice has discount factor \( \delta_A \), so \( \delta_B \) and \( \delta_A \) represent the level of impatience of the two players. For instance, if \( \delta_B > \delta_A \), then Bob does not get hurt as much by delaying agreement until the next rounds. We would expect that a more impatient player would get a small share of the pie in a subgame perfect Nash equilibrium and that is what we are about to see.

We must revise the extensive form game so that the payoffs are relative to the point at which the game ends, not relative to the beginning of the game. When it is Bob’s second time to offer, he will again offer \( x \) and if this is accepted, he will receive \( 1 - x \). If Alice is to induce Bob to accept her \( y \) offer on the previous round, we must have \( y \geq \delta_B(1 - x) \), so to maximize her payoff, Alice offers Bob \( y = \delta_B(1 - x) \). This means Alice gets \( 1 - y = 1 - \delta_B(1 - x) \). But then for Bob to induce Alice to accept on the first round, Bob must offer \( x \geq \delta_A(1 - y) \) and to maximize his payoff, he then sets \( x = \delta_A(1 - y) \). We then have

\[
x = \delta_A(1 - y) = \delta_A(1 - \delta_B(1 - x)),
\]

which gives

\[
x = \frac{\delta_A(1 - \delta_B)}{1 - \delta_B \delta_A}, \quad 1 - x = \frac{1 - \delta_A}{1 - \delta_B \delta_A}.
\]

To see what this means, let’s set \( \delta_A = 0.9 \) and vary \( \delta_B \) from 0 (Bob is infinitely impatient) to 1 (Bob is infinitely patient). We get figure 5.8
5.12 Bargaining with One Outside Option

Suppose in the Rubinstein bargaining model (§5.10) that if Alice rejects Bob’s offer, he receives an outside option of amount \( s_2 > 0 \) with probability \( p > 0 \), with no delay. If he accepts this option, he receives \( s_2 \), Bob receives nothing and the game is over.

To analyze this problem, let \( 1 - x \) be the maximum Bob can get in any subgame perfect Nash equilibrium, if we assume that Alice accepts the outside offer when it is available and that \( 1 - x \geq 0 \). Then, when it is Alice’s turn to offer, she must offer at least \( \delta(1 - x) \), so her maximum payoff when rejecting \( x \) is \( ps_2 + (1 - p)\delta(1 - \delta(1 - x)) \). The most Bob must offer Alice is this amount, so the most he can make satisfies the equation

\[
1 - x = 1 - (ps_2 + (1 - p)\delta(1 - \delta(1 - x))).
\]

which gives

\[
x = \frac{ps_2 + (1 - p)\delta(1 - \delta)}{1 - (1 - p)\delta^2}.
\]

A similar argument shows that \( x \) is the minimum Bob can get in a subgame perfect Nash equilibrium, if we assume that Alice accepts the outside offer when it is available and that \( 1 - x \geq 0 \). This shows that such an \( x \) is unique. Our assumption that Alice accepts the outside offer requires

\[
s_2 \geq \delta(1 - \delta(1 - x)),
\]
which is Alice’s payoff if she rejects the outside option. It is easy to show that this inequality holds exactly when \( s_2 \geq \delta/(1 + \delta) \). We also must have \( 1 - x \geq 0 \), or Bob will not offer \( x \). It is easy to show that this is equivalent to \( s_2 \leq (1 - \delta(1 - p))/p \).

It follows that if \( s_2 < \delta/(1 + \delta) \), there is a unique subgame perfect Nash equilibrium in which the players use the same strategies and receive the same payoffs, as if the outside option did not exist. In particular, Alice rejects the outside option when it is available. Moreover, if

\[
\frac{\delta}{1 + \delta} \leq s_2 \leq \frac{1 - (1 - p)\delta}{p},
\]

(5.1)

then there is a unique subgame perfect Nash equilibrium in which Alice would accept \( s_2 \) if it became available, but Bob offers Alice the amount

\[
x = \frac{ps_2 + (1 - p)\delta(1 - \delta)}{1 - (1 - p)^2 \delta^2},
\]

which Alice accepts.

Note that Bob’s payoff decreases from \( 1/(1 + \delta) \) to zero as \( s_2 \) increases over the interval (5.1).

Finally, if

\[
s_2 > \frac{1 - (1 - p)\delta}{p},
\]

then there is a subgame perfect Nash equilibrium in which Alice simply waits for the outside option to become available (that is, he accepts no offer \( x \leq 1 \) and offers a strictly negative amount).

### 5.13 Bargaining with Dual Outside Options

Alice and Bob bargain over splitting a dollar using the Rubinstein alternating-offer bargaining model with common discount factor \( \delta \) (§5.10). Alice makes the first offer. Bob, as respondent, can accept or reject the offer. If Bob rejects Alice’s offer, an outside option worth \( s_B > 0 \) becomes available to Bob with probability \( p > 0 \). If available, Bob can accept or reject the option. If Bob accepts the option, his payoff is \( s_B \) and Alice’s payoff is zero. Otherwise, the proposer and the respondent exchange roles and, after a delay of one time period, the game continues. Each game delay
decreases both the amount to be shared and the outside options by a factor of $\delta$.

The following reasoning specifies the unique subgame perfect Nash equilibria of this bargaining game. The game tree is depicted in figure 5.9, where $a$ means “accept” and $r$ means “reject.” Note that this is not a complete game tree, because we do not represent the players’ decisions concerning accepting or rejecting the outside offer.

First, if $s_A, s_B < \delta/(1 + \delta)$, then there is a unique subgame perfect Nash equilibrium in which Bob offers $\delta/(1 + \delta)$ and Alice accepts. To see this, note that under these conditions, the subgame perfect Nash equilibrium of the Rubinstein bargaining model without outside options is a subgame perfect Nash equilibrium of this game, with the added proviso that neither player takes the outside option when it is available. To see that rejecting the outside option is a best response, note that when Alice has the outside option, she also knows that if she rejects it, her payoff will be $\delta/(1 + \delta)$, so she should reject it. A similar argument holds for Bob.

Second, suppose the following inequalities hold:

$$p(1-p)\delta s_A - (1-p)\delta(1-(1-p)\delta) < s_B < p(1-p)\delta s_A + 1 - (1-p)\delta.$$  

$$s_A(1 - (1-p)\delta^2) + \delta s_B > \delta(1 - \delta(1-p)).$$  

Then, there is a subgame perfect Nash equilibrium in which Alice offers

$$x = \frac{ps_B + (1-p)\delta(1 - p s_A) - (1-p)^2\delta^2}{1 - (1-p)^2\delta^2}$$

and Bob accepts. In this subgame perfect Nash equilibrium, both players accept the outside option if it is available and Alice makes Bob an offer that
Bob accepts (note that the subgames at which these actions are carried out do not lie on the game path). Show also that in this case, $s_A$ and $s_B$ exist satisfying the preceding inequalities and all satisfy $s_A, s_B > \delta/(1 + \delta)$. To see this if we assume that both agents take the outside option when it is available, we have the recursive equation

$$x = ps_B + \delta(1 - p)(1 - (ps_A + (1 - p)\delta(1 - x))),$$

and the preceding inequalities ensure that $x \in [0, 1]$, that $s_A > \delta(1 - x)$, so Alice takes the outside option when available and $s_B > \delta(1 - ps_A - (1 - p)\delta(1 - x))$, so Bob takes the outside option when available. This justifies our assumption.

Third, suppose the following inequalities hold:

$$p(1 - p)\delta s_A - ps_B > (1 - p)\delta(1 - (1 - p)\delta),$$

$$ps_A + ps_B(1 - (1 - p)\delta) < 1 - \delta^2(1 - p)^2.$$  

Then, there is a subgame perfect Nash equilibrium in which Bob rejects Alice’s offer, Bob takes the outside option if it is available and if not, Bob offers Alice

$$\frac{ps_A}{1 - (1 - p)^2\delta^2}$$

and Alice accepts. In this subgame perfect Nash equilibrium, Alice also accepts the outside option if it is available. What are the payoffs to the two players? Show also that there are $s_A$ and $s_B$ that satisfy the preceding inequalities and we always have $s_B > \delta/(1 + \delta)$. To see this, first show that if Alice could make an acceptable offer to Bob, then the previous recursion for $x$ would hold, but now $x < 0$, which is a contradiction. Then, either Alice accepts an offer from Bob, or Alice waits for the outside option to become available. The payoff to waiting is

$$\pi_A = \frac{(1 - p)ps_A\delta}{1 - (1 - p)^2\delta^2},$$

but Alice will accept $\delta(ps_A + (1 - p)\delta\pi_A)$, leaving Bob with $1 - \delta(ps_A + (1 - p)\delta\pi_A)$. This must be better for Bob than just waiting for the outside option to become available, which has value, at the time Bob is the proposer,

$$\frac{(1 - p)ps_B\delta}{1 - (1 - p)^2\delta^2}.$$
Show that the preceding inequalities imply Bob will make an offer to Alice. Then, use the inequalities to show that Alice will accept. The remainder of the problem is now straightforward.

Show that this case applies to the parameters $\delta = 0.9$, $p = 0.6$, $s_A = 1.39$, and $s_B = 0.08$. Find Bob’s offer to Alice, calculate the payoffs to the players and show that Alice will not make an offer that Bob would accept.

Fourth, suppose

$$ps_A > 1 - (1 - p)^2 \delta > p(1 - p)\delta s_A + ps_B.$$  

Then, there is a subgame perfect Nash equilibrium where Alice offers Bob

$$\frac{ps_B}{1 - (1 - p)^2 \delta^2}$$

and Bob accepts. In this equilibrium, Alice accepts no offer and both players accept the outside option when it is available. It is easy to show also that there are $s_A$ and $s_B$ that satisfy the preceding inequalities and we always have $s_A > \delta / (1 + \delta)$. Note that in this case Alice must offer Bob

$$\pi_2 = \frac{ps_B}{1 - (1 - p)^2 \delta^2},$$

which is what Bob can get by waiting for the outside option. Alice minus this quantity must be greater than $\pi_A$, or Alice will not offer it. Show that this holds when the preceding inequalities hold. Now, Alice will accept $ps_A + (1 - p)\delta \pi_A$, but you can show that this is greater than 1, so Bob will not offer this. This justifies our assumption that Alice will not accept anything that Bob is willing to offer.

This case applies to the parameters $\delta = 0.9$, $p = 0.6$, $s_A = 1.55$, and $s_B = 0.88$. Find Alice’s offer to Bob, calculate the payoffs to the players and show that Bob will not make an offer that Alice would accept.

Finally, suppose the following inequalities hold:

$$1 - \delta^2 (1 - p)^2 < ps_B + ps_A \delta (1 - p),$$

$$1 - \delta^2 (1 - p)^2 < ps_B + ps_A \delta (1 - p).$$

Then, the only subgame perfect Nash equilibrium has neither agent making an offer acceptable to the other, so both agents wait for the outside option to become available. Note that by refusing all offers and waiting for the
outside option to become available, Alice’s payoff is $\pi_A$ and Bob’s payoff is $\pi_2$. Show that the inequalities imply $1 - \pi_2 < \pi_A$, so Alice will not make an acceptable offer to Bob and $1 - \pi_A/\delta(1 - p) < \delta(1 - p)\pi_2$, so Bob will not make an acceptable offer to Alice.

5.14 Huey, Dewey, and Louie Split a Dollar

Huey, Dewey, and Louie have a dollar to split. Huey gets to offer first, and offers shares $d$ and $l$ to Dewey and Louie, keeping $h$ for himself (so $h + d + l = 1$). If both accept, the game is over and the dollar is divided accordingly. If either Dewey or Louie rejects the offer, however, they come back the next day and start again, this time Dewey making the offer to Huey and Louie and if this is rejected, on the third day Louie gets to make the offer. If this is rejected, they come back on the fourth day with Huey again making the offer. This continues until an offer is accepted by both players, or until the universe winks out, in which case they get nothing. However, the present value of a dollar tomorrow for each of the players is $\delta < 1$.

There exists a unique symmetric (that is, all players use the same strategy), stationary (that is, players make the same offers, as a fraction of the pie, on each round), subgame perfect equilibrium (interestingly enough, there exist other, nonsymmetric subgame perfect equilibria, but they are difficult to describe and could not occur in the real world).

The game tree for this problem appears in figure 5.10, where the equilibrium shares are $(h, d, l)$. We work back the game tree (which is okay, because we are looking only for subgame perfect equilibria). At the second place where Huey gets to offer (at the right side of the game tree), the
value of the game to Huey is $h$, because we assume a stationary equilibrium. Thus, Louie must offer Huey at least $\delta h$ where Louie gets to offer, to get Huey to accept. Similarly, Louie must offer Dewey at least $\delta d$ at this node. Thus, the value of the game where Louie gets to offer is $(1 - \delta h - \delta d)$.

When Dewey gets to offer, he must offer Louie at least $\delta$ times what Louie gets when it is Louie’s turn to offer, to get Louie to accept. This amount is just $\delta(1 - \delta h - \delta d)$. Similarly, he must offer Huey $\delta^2 h$ to accept, because Huey gets $\delta h$ when it is Louie’s turn to offer. Thus, Dewey gets

$$1 - \delta(1 - \delta h - \delta d) - \delta^2 h = 1 - \delta(1 - \delta d)$$

when it is his turn to offer.

Now Huey, on his first turn to offer, must offer Dewey $\delta$ times what Dewey can get when it is Dewey’s turn to offer, or $\delta(1 - \delta(1 - \delta d))$. But then we must have

$$d = \delta(1 - \delta(1 - \delta d)).$$

Solving this equation for $d$, we find

$$d = \frac{\delta}{1 + \delta + \delta^2}.$$

Moreover, Huey must offer Louie $\delta$ times what Dewey would offer Louie in the next period or $\delta^2 (1 - \delta h - \delta d)$. Thus, Huey offers Dewey and Louie together

$$\delta(1 - \delta(1 - \delta d)) + \delta^2 (1 - \delta h - \delta d) = \delta - \delta^3 h,$$

so Huey gets $1 - \delta + \delta^3 h$ and this must equal $h$. Solving, we get

$$h = \frac{1}{1 + \delta + \delta^2},$$

so we must have

$$l = 1 - d - h = \frac{\delta^2}{1 + \delta + \delta^2},$$

which is the solution to the problem.

Note that there is a simpler way to solve the problem, just using the fact that the solution is symmetric: we must have $d = \delta h$ and $l = \delta d$, from which the result follows. This does not make clear, however, where subgame perfection comes in.
5.15 Twin Sisters

A mother has twin daughters who live in different towns. She tells each to ask for a certain whole number of dollars, at least 1 and at most 100, as a birthday present. If the total of the two amounts does not exceed 101, each will have her request granted. Otherwise each gets nothing.

a. Find all the pure-strategy Nash equilibria of this game.
b. Is there a symmetric equilibrium among these? A symmetric Nash equilibrium is one in which both players use the same strategy.
c. What do you think the twins will most likely do, if we assume that they cannot communicate? Why? Is this a Nash equilibrium?

5.16 The Samaritan’s Dilemma

Many conservatives dislike Social Security and other forms of forced saving by means of which the government prevents people from ending up in poverty in their old age. Some liberals respond by claiming that people are too short-sighted to manage their own retirement savings successfully. James Buchanan (1975) has made the insightful point that even if people are perfectly capable of managing their retirement savings, if we are altruistic toward others, we will force people to save more than they otherwise would.3 Here is a simple model exhibiting his point.

A father and a daughter have current income \( y > 0 \) and \( z > 0 \), respectively. The daughter saves an amount \( s \) for her schooling next year and receives an interest rate \( r > 0 \) on her savings. She also receives a transfer \( t \) from her father in the second period. Her utility function is

\[
v(s, t) = v_A(z - s) + \delta v_2(s(1 + r) + t),
\]

where \( \delta > 0 \) is a discount factor, \( v_A \) is her first-period utility and \( v_2 \) is her second-period utility. Her father has personal utility function \( u(y) \), but he has degree \( \alpha > 0 \) of altruistic feeling for his daughter, so he acts to maximize \( U = u(y - t) + \alpha v(s, t) \). Suppose all utility functions are increasing and concave, the daughter chooses \( s \) to maximize her utility, the father observes the daughter’s choice of \( s \) and then chooses \( t \). Let \((s^*, t^*)\) be the resulting equilibrium. Show that the daughter will save too little, in the sense that if the father can precommit to \( t^* \), both she and her father would be better off. Show by example that, if we assume

3In this case, by an altruist we mean an agent who takes actions that improve the material well-being of other agents at a material cost to himself.
that $u(\cdot) = v_A(\cdot) = v_2(\cdot) = \ln(\cdot)$ and that the father can precommit, he may precommit to an amount less than $t^*$. Note first that the father’s first-order condition is

$$U_t(t, s) = -u'(y - t) + \alpha \delta v_2'(s(1 + r) + t) = 0,$$

and the father’s second-order condition is

$$U_{tt} = u''(y - t) + \alpha \delta v_2''(s(1 + r) + t) < 0.$$

If we treat $t$ as a function of $s$ (one step of backward induction, which is uncontroversial, because each player moves only once), then the equation $U_t(t(s), s) = 0$ is an identity, so we can differentiate it totally with respect to $s$, getting

$$U_{tt} \frac{dt}{ds} + U_{ts} = 0.$$

But $U_{ts} = \alpha \delta (1 + r)v_2'' < 0$, so $t'(s) < 0$; that is, the more the daughter saves, the less she gets from her father in the second period.

Now, the daughter’s first-order condition is

$$v_s(s, t) = -v_A' + \delta v_2'(1 + r + t'(s)) = 0.$$

Suppose the daughter’s optimal $s$ is $s^*$, so the father’s transfer is $t^* = t(s^*)$. If the father precommits to $t^*$, then $t'(s) = 0$ would hold in the daughter’s first-order condition. Therefore, in this case $v_s(s^*, t^*) > 0$, so the daughter is better off by increasing $s$ to some $s^{**} > s^*$. Thus, the father is better off as well, because he is a partial altruist.

For the example, if $u(\cdot) = v_A(\cdot) = v_2(\cdot) = \ln(\cdot)$, then it is straightforward to check that

$$t^* = \frac{y(1 + \alpha \delta (1 + \delta)) - \delta(1 + r)z}{(1 + \delta)(1 + \alpha \delta)},$$

$$s^* = \frac{\delta(1 + r)z - y}{(1 + r)(1 + \delta)}.$$

If the father can precommit, solving the two first-order conditions for maximizing $U(t, s)$ gives

$$t^f = \frac{\alpha(1 + \delta)y - (1 + r)z}{1 + \alpha + \alpha \delta},$$

$$s^f = \frac{(1 + r)(1 + \alpha \delta)z - \alpha y}{(1 + r)(1 + \alpha + \alpha \delta)}.$$
We then find
\[
\frac{t^* - t^f}{ETX} = \frac{y + (1 + r)z}{(1 + \delta)(1 + \alpha \delta)(1 + \alpha + \alpha \delta)} > 0,
\]
\[
\frac{s^f - s^*}{NUL} = \frac{y + (1 + r)z}{(1 + r)(1 + \delta)(1 + \alpha + \alpha \delta)} > 0.
\]

### 5.17 The Rotten Kid Theorem

This problem is the core of Gary Becker’s (1981) famous theory of the family. You might check the original, though, because I’m not sure I got the genders right.

A certain family consists of a mother and a son, with increasing, concave utility functions \( u(y) \) for the mother and \( v(z) \) for the son. The son can affect both his income and that of the mother by choosing a level of familial work commitment \( a \), so \( y = y(a) \) and \( z = z(a) \). The mother, however, feels a degree of altruism \( \alpha > 0 \) toward the son, so given \( y \) and \( z \), she transfers an amount \( t \) to the son to maximize the objective function

\[
V(t; a) = u(y(a) - t) + \alpha v(z(a) + t).
\]

The son, however, is perfectly selfish (“rotten”) and chooses the level of \( a \) to maximize his own utility \( v(z(a) + t) \). However, he knows that his mother’s transfer \( t \) depends on \( y \) and \( z \) and hence on \( a \).

We will show that the son chooses \( a \) to maximize total family income \( y(a) + z(a) \) and \( t \) is an increasing function of \( a \). Also, if we write \( y = y(a) + \hat{y} \), then \( t \) is an increasing function of the mother’s exogenous wealth \( \hat{y} \). We can also show that for sufficiently small \( \alpha > 0, t < 0 \); that is, the transfer is from the son to the mother.

First, Mom’s objective function is

\[
V(t, a) = u(y(a) - t) + \alpha v(z(a) + t),
\]

so her first-order condition is

\[
V_t(t, a) = -u'(y(a) - t) + \alpha v'(z(a) + t) = 0.
\]

If we treat \( t \) as a function of \( a \) in the preceding equation (this is one stage of backward induction, which is uncontroversial), it becomes an identity, so we can differentiate with respect to \( a \), getting

\[
-u''(y' - t') + \alpha v''(z' + t') = 0.
\]
Therefore, $z' + t' = 0$ implies $y' - t' = y' + z' = 0$. Thus, the first-order conditions for the maximization of $z + t$ and $z + y$ have the same solutions.

Note that because $a$ satisfies $z'(a) + y'(a) = 0$, $a$ does not change when $\alpha$ changes. Differentiating the first-order condition $V_t(t(\alpha)) = 0$ totally with respect to $\alpha$, we get

$$V_t \frac{dt}{\alpha} + V_{t\alpha} = 0.$$ 

Now $V_{tt} < 0$ by the second-order condition for a maximum and

$$V_{t\alpha} = v' > 0,$$

which proves that $dt/d\alpha > 0$. Because $a$ does not depend on $\hat{y}$, differentiating $V_t(t(y)) = 0$ totally with respect to $\hat{y}$, we get

$$V_t \frac{dt}{\hat{y}} + V_{t\hat{y}} = 0.$$ 

But $V_{t\hat{y}} = -u'' > 0$ so $dt/d\hat{y} > 0$.

Now suppose $t$ remains positive as $\alpha \to 0$. Then $v'$ remains bounded, so $\alpha v' \to 0$. From the first-order condition, this means $u' \to 0$, so $y - t \to \infty$. But $y$ is constant, because $a$ maximizes $y + z$, which does not depend on $\alpha$. Thus $t \to -\infty$.

### 5.18 The Shopper and the Fish Merchant

A shopper encounters a fish merchant. The shopper looks at a piece of fish and asks the merchant, “Is this fish fresh?” Suppose the fish merchant knows whether the fish is fresh or not and the shopper knows only that the probability that any particular piece of fish is fresh is 1/2. The merchant can then answer the question either yes or no. The shopper, upon hearing this response, can either buy the fish or wander on.

Suppose both parties are risk neutral (that is, they have linear utility functions and hence maximize the expected value of lotteries), with utility functions $u(x) = x$, where $x$ is an amount of money. Suppose the price of the fish is 1, the value of a fresh fish to the shopper is 2 (that is, this is the maximum the shopper would pay for fresh fish) and the value of fish that is not fresh is zero. Suppose the fish merchant must throw out the fish if it is not sold, but keeps the 1 profit if she sells the fish. Finally, suppose the
The extensive form for the game is shown in figure 5.11 and the normal form for each of the two cases good fish/bad fish and their expected value is given in figure 5.12. Applying the successive elimination of dominated strategies, we have $yn$ dominates $ny$, after which $bn$ dominates $bb$, $nb$, and $nn$. But then $yy$ dominates $yn$ and $nn$. Thus, a Nash equilibrium is
In this game, the merchant says the fish is good no matter what and the buyer believes him. Because some of the eliminated strategies were only weakly dominated, there could be other Nash equilibria and we should check for this. We find that another is for the seller to use pure strategy $nn$ and the buyer to use pure strategy $nb$. Note that this equilibrium works only if the buyer is a “nice guy” in the sense of choosing among equally good responses that maximizes the payoff to the seller. The equilibrium $yy/bn$ does not have this drawback.

### 5.19 Pure Coordination Games

We say one allocation of payoffs Pareto-dominates another, or is Pareto-superior to another, if all players are at least as well off in the first as in the second and at least one is better off. We say one allocation of payoffs strictly Pareto-dominates another if all players are strictly better off in the first than in the second. We say an allocation is Pareto-efficient or Pareto-optimal if it is not Pareto-dominated by any other allocation. An allocation is Pareto-inefficient if it is not Pareto-efficient. A pure coordination game is a game in which there is one pure-strategy Nash equilibrium that strictly Pareto-dominates all other Nash equilibria.

a. Consider the game where you and your partner independently guess an integer between 1 and 5. If you guess the same number, you each win the amount of your guess. Otherwise you lose the amount of your guess. Show that this is a pure coordination game. Hint: Write down the normal form of this game and find the pure-strategy Nash equilibria.

b. Consider a two-player game in which each player has two strategies. Suppose the payoff matrices for the two players are $a_{ij}$ for player 1 and $b_{ij}$ for player 2, where $i,j = 1,2$. Find the conditions on these payoff matrices for the game to be pure coordination game. Hint: First solve this for a $2 	imes 2$ game, then a $3 	imes 3$, then generalize.

### 5.20 Pick Any Number

Three people independently choose an integer between zero and nine. If the three choices are the same, each person receives the amount chosen. Otherwise each person loses the amount the person chose.

a. What are the pure-strategy Nash equilibria of this game?
b. How do you think people will actually play this game?

c. What does the game look like if you allow communication among the players before they make their choices? How would you model such communication and how do you think communication would change the behavior of the players?

5.21 Pure Coordination Games: Experimental Evidence

<table>
<thead>
<tr>
<th>Your Choice of $x$</th>
<th>Smallest value of $x$ chosen (including own)</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>1.30</td>
</tr>
<tr>
<td>6</td>
<td>1.20</td>
</tr>
<tr>
<td>5</td>
<td>1.10</td>
</tr>
<tr>
<td>4</td>
<td>1.00</td>
</tr>
<tr>
<td>3</td>
<td>0.90</td>
</tr>
<tr>
<td>2</td>
<td>0.80</td>
</tr>
<tr>
<td>1</td>
<td>0.70</td>
</tr>
</tbody>
</table>

Table 5.1. An experimental coordination game (Van Huyck, Battalio, and Beil, 1990)

Show that the game in table 5.1 is a pure coordination game and find the number of pure-strategy equilibria. In this game, a number $n$ of players each chooses a number between 1 and 7. Suppose $x_i$ is the choice of player $i$ and the lowest of the numbers is $y$. Then player $i$ wins $0.60 + 0.10 \times x_i - 0.20 \times (x_i - y)$.

Most people think it is obvious that players will always play the Pareto-optimal $x_i = 7$. As van Huyck, Battalio, and Beil (1990) show, this is far from the case. The experimenters recruited 107 Texas A&M undergraduates and the game was played ten times with $n$ varying between 14 and 16 subjects. The results are shown in table 5.2. Note that in the first round, only about 30% of the subjects chose the Pareto-efficient $x = 7$ and because the lowest choice was $y = 1$, they earned only 0.10. Indeed, the subjects who earned the most were precisely those who chose $x = 1$. The subjects progressively learn from trial to trial that it is hopeless to choose a high number and in the last round almost all subjects are choosing $x = 1$ or $x = 2$. 
Table 5.2. Results of ten trials with 107 Subjects. Each entry represents the number of subjects who chose \( x \) (row) in period \( y \) (column).

<table>
<thead>
<tr>
<th>Choice of ( x )</th>
<th>Period</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>7</td>
<td>33</td>
</tr>
<tr>
<td>6</td>
<td>10</td>
</tr>
<tr>
<td>5</td>
<td>34</td>
</tr>
<tr>
<td>4</td>
<td>17</td>
</tr>
<tr>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>

Why do people do such a poor job of coordinating in situations like these? A possibility is that not all subjects really want to maximize their payoff. If one subject wants to maximize his payoff relative to the other player, then \( x = 1 \) is the optimal choice. Moreover, if one or more players think that there might be a player who is such a “relative maximizer,” such players will play \( x = 1 \) even if they want to maximize their absolute payoffs. There are also other possible explanations, such as wanting to maximize the minimum possible payoff.

### 5.22 Introductory Offers

A product comes in two qualities, high and low, at unit costs \( c_h \) and \( c_l \), with \( c_h > c_l > 0 \). Consumers purchase one unit per period and a consumer learns the quality of a firm’s product only by purchasing it in the first period. Consumers live for two periods and a firm cannot change its quality between the first and second periods. Thus, a consumer can use the information concerning product quality gained in the first period to decide whether to buy from the firm again in the second period. Moreover, firms can discriminate between first- and second-period consumers and offer different prices in the two periods, for instance, by extending an introductory offer to a new customer.

Suppose the value of a high-quality good to the consumer is \( h \), the value of a low-quality good is zero, a consumer will purchase the good only if this does not involve a loss and a firm will sell products only if it makes positive profits. We say that the industry is in a truthful signaling equilibrium if
the firms’ choice of sale prices accurately distinguishes high-quality from low-quality firms. If the firms’ choices do not distinguish high from low quality, we have a pooling equilibrium. In the current situation, this means that only the high-quality firms will sell.

Let \( \delta \) be the consumer’s discount factor on second-period utility. We show that if \( h > c_h + (c_h - c_l)/\delta \), there is a truthful signaling equilibrium and not otherwise. If a high-quality firm sells to a consumer in the first period at some price \( p_1 \), then in the second period the consumer will be willing to pay \( p_2 = h \), because he knows the product is of high quality. Knowing that it can make a profit \( h - c_h \) from a customer in the second period, a high-quality firm might want to make a consumer an “introductory offer” at a price \( p_1 \) in the first period that would not be mimicked by the low-quality firm, in order to reap the second-period profit.

If \( p_1 > c_l \), the low-quality firm could mimic the high-quality firm, so the best the high-quality firm can do is to charge \( p_1 = c_l \), which the low-quality firm will not mimic, because the low-quality firm cannot profit by doing so (it cannot profit in the first period and the consumer will not buy the low-quality product in the second period). In this case, the high-quality firm’s profits are \( (c_l - c_h) + \delta(h - c_h) \). As long as these profits are positive, which reduces to \( h > c_h + \delta(c_h - c_l) \), the high-quality firm will stay in business.

Note that each consumer gains \( h - c_l \) in the truthful signaling equilibrium and firms gain \( c_l - c_h + \delta(h - c_h) \) per customer.

### 5.23 Web Sites (for Spiders)

In the spider *Agelenopsis aperta*, individuals search for desirable locations for spinning webs. The value of a web is \( 2v \) to its owner. When two spiders come upon the same desirable location, the two invariably compete for it. Spiders can be either strong or weak, but it is impossible to tell which type a spider is by observation. A spider may rear onto two legs to indicate that it is strong, or fail to do so, indicating that it is weak. However, spiders do not have to be truthful. Under what conditions will they in fact signal truthfully whether they are weak or strong? Note that if it is in the interest of both the weak and the strong spider to represent itself as strong, we have a “pooling equilibrium,” in which the value of the signal is zero and it will be totally ignored; hence, it will probably not be issued. If only the strong spider signals, we have a truthful signaling equilibrium.
Assume that when two spiders meet, each signals the other as strong or weak. Sender sends a signal to Receiver, Receiver simultaneously sends a signal to Sender and they each choose actions simultaneously. Based on the signal, each spider independently decides to attack or retreat. If two strong spiders attack each other, they each incur a cost of $c_s$ and each has a 50% chance of gaining or keeping the territory. Thus, the expected payoff to each is $v - c_s$. If both spiders retreat, neither gets the territory, so their expected payoff is 0 for each. If one spider attacks and the other retreats, the attacker takes the location and there are no costs. So the payoffs to attacker and retreater are $2v$ and 0, respectively. The situation is the same for two weak spiders, except they have a cost $c_w$. If a strong and a weak spider attack each other, the strong wins with probability 1, at a cost $b$ with $c_s > b > 0$ and the weak spider loses, at a cost $d > 0$. Thus, the payoff to the strong spider against the weak is $2v - b$ and the payoff to the weak against the strong is $-d$. In addition, strong spiders incur a constant cost per period of $e$ to maintain their strength. Figure 5.13 shows a summary of the payoffs for the game.

<table>
<thead>
<tr>
<th>Type 1, Type 2</th>
<th>Action 1, Action 2</th>
<th>Payoff 1, Payoff 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>strong, strong</td>
<td>attack, attack</td>
<td>$v - c_s, v - c_s$</td>
</tr>
<tr>
<td>weak, weak</td>
<td>attack, attack</td>
<td>$v - c_w, v - c_w$</td>
</tr>
<tr>
<td>strong, weak</td>
<td>attack, attack</td>
<td>$2v - b, -d$</td>
</tr>
<tr>
<td>either, either</td>
<td>attack, retreat</td>
<td>$2v, 0$</td>
</tr>
<tr>
<td>either, either</td>
<td>retreat, retreat</td>
<td>$0, 0$</td>
</tr>
</tbody>
</table>

Figure 5.13. Web sites for spiders

Each spider has eight pure strategies: signal that it is strong or weak ($s/w$), attack/retreat if the other spider signals strong ($a/r$), attack/retreat if the other spider signals weak ($a/r$). We may represent these eight strategies as $saa$, $sar$, $sra$, $srr$, $waa$, $war$, $wra$, $wrr$, where the first indicates the spider’s signal, the second indicates the spider’s move if the other spider signals strong, and the third indicates the spider’s move if the other spider signals weak (for instance, $sra$ means “signal strong, retreat from a strong signal, and attack a weak signal”). This is a complicated game, because the payoff matrix for a given pair of spiders has 64 entries and there are four types of pairs of spiders. Rather than use brute force, let us assume
there is a truthful signaling equilibrium and see what that tells us about the relationships among \( v, b, c_w, c_s, d, e \) and the fraction \( p \) of strong spiders in the population.

Suppose \( v > c_s, c_w \) and the proportion \( p \) of strong spiders is determined by the condition that the payoffs to the two conditions of being strong and being weak are equal. In a truthful signaling equilibrium, strong spiders use \( saa \) and weak spiders use \( wra \). To see this, note that strong spiders say they’re strong and weak spiders say they’re weak, by definition of a truthful signaling equilibrium. Weak spiders retreat against strong spiders because \( d > 0 \) and attack other weak spiders because \( v - c_w > 0 \). Strong spiders attack weak spiders if they do not withdraw, because \( 2v - b > 2v - c_s > v \).

If \( p \) is the fraction of strong spiders, then the expected payoff to a strong spider is \( p(v - c_s) + 2(1 - p)v - e \) and the expected payoff to a weak spider is \( (1 - p)(v - c_w) \). If these two are equal, then

\[
p = \frac{v + c_w - e}{c_w + c_s}, \tag{5.4}
\]

which is strictly between 0 and 1 if and only if \( e - c_w < v < e + c_s \).

In a truthful signaling equilibrium, each spider has expected payoff

\[
\pi = \frac{(v - c_w)(c_s + e - v)}{c_w + c_s}. \tag{5.5}
\]

Suppose a weak spider signals that it is strong and all other spiders play the truthful signaling equilibrium strategy. If the other spider is strong, it will attack and the weak spider will receive \(-d\). If the other spider is weak, it will withdraw and the spider will gain \(2v\). Thus, the payoff to the spider for a misleading communication is \(-pd + 2(1 - p)v\), which cannot be greater than (5.5) if truth telling is Nash. Solving for \( d \), we get

\[
d \geq \frac{(c_s + e - v)(v + c_w)}{c_w - e + v}.
\]

Can a strong spider benefit from signaling that it is weak? To see that it cannot, suppose first that it faces a strong spider. If it attacks the strong spider after signaling that it is weak, it gets the same payoff as if it signaled strong (because its opponent always attacks). If it withdraws against its opponent, it gets 0, which is less than the \( v - c_s \) it gets by attacking. Thus, signaling weak against a strong opponent cannot lead to a gain. Suppose
the opponent is weak. Then signaling weak means that the opponent will
attack. Responding by withdrawing, it gets 0; responding by attacking, it
gets $2v - b$, because it always defeats its weak opponent. But if it had
signaled strong, it would have earned $2v > 2v - b$. Thus, it never pays a
strong spider to signal that it is weak.

Note that as long as both strong and weak spiders exist in equi-
librium, an increase in the cost $e$ of being strong leads to an increase in payoff
to all spiders, weak and strong alike. This result follows directly from
equation (5.5) and is due to the fact that higher $e$ entails a lower fraction
of strong spiders, from (5.4). But weak spiders earn $(1 - p)(v - c_w)$,
which is decreasing in $p$ and strong spiders earn the same as weak spiders
in equilibrium.