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## Markov Economies and Stochastic Dynamical Systems

God does not play dice  
with the Universe.

Albert Einstein

Time-discrete stochastic processes are powerful tools for characterizing some dynamical systems. The prerequisites include an understanding of Markov chains (§13.1). Time-discrete systems behave quite differently from dynamical systems based on systems of ordinary differential equations. This chapter presents a Markov model of adaptive learning that illustrates the concept of stochastic stability, as developed in Young (1998). After developing some of the theoretical results, we provide an agent-based model.

### 13.1 Markov Chains

A *finite Markov chain* is a dynamical system that in each time period  $t = 0, 1, \dots$  can be any one of  $n$  states, such that if the system is in state  $i$  in one time period, there is a probability  $p_{ij}$  that the system will be in state  $j$  in the next time period. Thus, for each  $i$ , we must have  $\sum_j p_{ij} = 1$ , because the system must go somewhere in each period. We call the  $n \times n$  matrix  $P = \{p_{ij}\}$  the *transition probability matrix* of the Markov chain, and each  $p_{ij}$  is called a *transition probability*. A *denumerable Markov chain* has an infinite number of states  $t = 1, 2, \dots$ , and is otherwise the same. If we do not care whether the finite or denumerable case obtains, we speak simply of a *Markov chain*.

Many games can be viewed as Markov chains. Here are some examples:

- a. Suppose two gamblers have wealth  $k_1$  and  $k_2$  dollars, respectively, and in each period they play a game in which each has an equal chance of winning one dollar. The game continues until one player has no more wealth. Here the state of the system is the wealth  $w$  of player 1,  $p_{w,w+1} = p_{w,w-1} = 1/2$  for  $0 < w < k_1 + k_2$ , and all other transition probabilities are zero.

- b. Suppose  $n$  agents play a game in which they are randomly paired in each period, and the stage game is a prisoner's dilemma. Players can remember the last  $k$  moves of their various partners. Players are also given one of  $r$  strategies, which determine their next move, depending on their current histories. When a player dies, which occurs with a certain probability, it is replaced by a new player who is a clone of a successful player. We can consider this a Markov chain in which the state of the system is the history, strategy, and score of each player, and the transition probabilities are just the probabilities of moving from one such state to another, given the players' strategies (§13.8).
- c. Suppose  $n$  agents play a game in which they are randomly paired in each period to trade. Each agent has an inventory of goods to trade and a strategy indicating which goods the agent is willing to trade for which other goods. After trading, agents consume some of their inventory and produce more goods for their inventory, according to some consumption and production strategy. When an agent dies, it is replaced by a new agent with the same strategy and an empty inventory. If there is a maximum-size inventory and all goods are indivisible, we can consider this a finite Markov chain in which the state of the system is the strategy and inventory of each player and the transition probabilities are determined accordingly.
- d. In a population of beetles, females have  $k$  offspring in each period with probability  $f_k$ , and beetles live for  $n$  periods. The state of the system is the fraction of males and females of each age. This is a denumerable Markov chain, where the transition probabilities are calculated from the birth and death rates of the beetles.

We are interested in the long-run behavior of Markov chains. In particular, we are interested in the behavior of systems that we expect will attain a long-run equilibrium of some type independent from its initial conditions. If such an equilibrium exists, we say the Markov chain is *ergodic*. In an ergodic system, history does not matter: every initial condition leads to the same long-run behavior. Nonergodic systems are history dependent. It is intuitively reasonable that the repeated prisoner's dilemma and the trading model described previously are ergodic. The gambler model is not ergodic,

because the system could end up with either player bankrupt.<sup>1</sup> What is your intuition concerning the beetle population, if there is a positive probability that a female has no offspring in a breeding season?

It turns out that there is a very simple and powerful theorem that tells us exactly when a Markov chain is ergodic and provides a simple characterization of the long-run behavior of the system. To develop the machinery needed to express and understand this theorem, we will define a few terms. Let  $p_{ij}^{(m)}$  be the probability of being in state  $j$  in  $m$  periods if the chain is currently in state  $i$ . Thus, if we start in state  $i$  at period 1, the probability of being in state  $j$  at period 2 is just  $p_{ij}^{(1)} = p_{ij}$ . To be in state  $j$  in period 3 starting from state  $i$  in period 1, the system must move from state  $i$  to some state  $k$  in period 2, and then from  $k$  to  $j$  in period 3. This happens with probability  $p_{ik}p_{kj}$ . Adding up over all  $k$ , the probability of being in state  $j$  in period 3 is

$$p_{ij}^{(2)} = \sum_k p_{ik} p_{kj}.$$

Using matrix notation, this means the matrix of two-period transitions is given by

$$P^{(2)} = \{p_{ij}^{(2)} | i, j = 1, 2, \dots\} = P^2.$$

Generalizing, we see that the  $k$ -period transition matrix is simply  $P^k$ . What we are looking for, then, is the limit of  $P^k$  as  $k \rightarrow \infty$ . Let us call this limit (supposing it exists)  $P^* = \{p_{ij}^*\}$ . Now  $P^*$  must have two properties. First, because the long-run behavior of the system cannot depend on where it started,  $p_{ij}^* = p_{i'j}^*$  for any two states  $i$  and  $i'$ . This means that all the rows of  $P^*$  must be the same. Let us denote the (common value of the) rows by  $u = \{u_1, \dots, u_n\}$ , so  $u_j$  is the probability that the Markov chain will be in state  $j$  in the long run. The second fact is that

$$PP^* = P \lim_{k \rightarrow \infty} P^k = \lim_{k \rightarrow \infty} P^{k+1} = P^*.$$

This means  $u$  must satisfy

$$u_j = \lim_{m \rightarrow \infty} p_{ij}^{(m)} \quad \text{for } i = 1, \dots, n \quad (13.1)$$

<sup>1</sup>Specifically, you can show that the probability that player 1 wins is  $k_1/(k_1 + k_2)$ , and if player 1 has wealth  $w$  at some point in the game, the probability he will win is  $w/(k_1 + k_2)$ .

$$u_j = \sum_i u_i p_{ij} \quad (13.2)$$

$$\sum_k u_k = 1, \quad (13.3)$$

for  $j = 1, \dots, n$ . Note that (13.2) can be written in matrix notation as  $u = uP$ , so  $u$  is a *left eigenvector* of  $P$ . The first equation says that  $u_j$  is the limit probability of being in state  $j$  starting from any state, the second says that the probability of being in state  $j$  is the probability of moving from some state  $i$  to state  $j$ , which is  $u_i p_{ij}$ , summed over all states  $i$ , and the final equation says  $u$  is a probability distribution over the states of the Markov chain. The *recursion equations* (13.2) and (13.3) are often sufficient to determine  $u$ , which we call the *invariant distribution* or *stationary distribution* of the Markov chain.

In the case where the Markov chain is finite, the preceding description of the stationary distribution is a result of the *Frobenius-Perron* theorem (Horn and Johnson 1985), which says that  $P$  has a maximum eigenvalue of unity, and the associated left eigenvector, which is the stationary distribution  $(u_1, \dots, u_n)$  for  $P$ , exists and has nonnegative entries. Moreover, if  $P^k$  is strictly positive for some  $k$  (in which case we say  $P$  is irreducible), then the stationary distribution has strictly positive entries.

In case a Markov chain is not ergodic, it is informative to know the whole matrix  $P^* = (p_{ij}^*)$ , because  $p_{ij}$  tell you the probability of being absorbed by state  $j$ , starting from state  $i$ . The Frobenius-Perron theorem is useful here also, because it tells us that all the eigenvalues of  $P$  are either unity or strictly less than unity in absolute value. Thus, if  $D = (d_{ij})$  is the  $n \times n$  diagonal matrix with the eigenvalues of  $P$  along the diagonal, then  $D^* = \lim_{k \rightarrow \infty} D^k$  is the diagonal matrix with zeros everywhere except unity where  $d_{ii} = 1$ . But, if  $M$  is the matrix of left eigenvectors of  $P$ , then  $MPM^{-1} = D$ , which follows from the definitions, implies  $P^* = M^{-1}D^*M$ . This equation allows us to calculate  $P^*$  rather easily.

A few examples are useful to get a feel for the recursion equations. Consider first the  $n$ -state Markov chain called the *random walk on a circle*, in which there are  $n$  states, and from any state  $t = 2, \dots, n - 1$  the system moves with equal probability to the previous or the next state, from state  $n$  it moves with equal probability to state 1 or state  $n - 1$ , and from state 1 it moves with equal probability to state 2 and to state  $n$ . In the long run, it is intuitively clear that the system will be all states with equal probability

$1/n$ . To derive this from the recursion equations, note that the probability transition matrix for the problem is given by

$$P = \begin{bmatrix} 0 & 1/2 & 0 & \dots & 0 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & \dots & 0 & 0 & 0 \\ & & & \vdots & & & \\ 0 & 0 & 0 & \dots & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 0 & \dots & 0 & 1/2 & 0 \end{bmatrix}.$$

The recursion equations for this system are given by

$$u_1 = \frac{1}{2}u_n + \frac{1}{2}u_2$$

$$u_i = \frac{1}{2}u_{i-1} + \frac{1}{2}u_{i+1} \quad i = 2, \dots, n-1$$

$$u_n = \frac{1}{2}u_1 + \frac{1}{2}u_{n-1}$$

$$\sum_{i=1}^n u_i = 1.$$

Clearly, this set of equations has solution  $u_i = 1/n$  for  $i = 1, \dots, n$ . Prove that this solution is unique by showing that if some  $u_i$  is the largest of the  $\{u_k\}$ , then its neighbors are equally large.

Consider next a closely related  $n$ -state Markov chain called the *random walk on the line with reflecting barriers*, in which from any state  $2, \dots, n-1$  the system moves with equal probability to the previous or the next state, but from state 1 it moves to state 2 with probability 1, and from state  $n$  it moves to state  $n-1$  with probability 1. Intuition in this case is a bit more complicated, because states 1 and  $n$  behave differently from the other states. The probability transition matrix for the problem is given by

$$P = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & \dots & 0 & 0 & 0 \\ & & & \vdots & & & \\ 0 & 0 & 0 & \dots & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 \end{bmatrix}.$$

The recursion equations for this system are given by

$$\begin{aligned} u_1 &= u_2/2 \\ u_i &= u_{i-1}/2 + u_{i+1}/2 \quad i = 2, \dots, n-1 \\ u_n &= u_{n-1}/2 \\ \sum_{i=1}^n u_i &= 1. \end{aligned}$$

It is easy to check directly that  $u_i = 1/(n-1)$  for  $i = 2, \dots, n-1$ , and  $u_1 = u_n = 1/2(n-1)$ . In fact, there is a general method for solving difference equations of this type, as described in section 13.6.

We can use the same methods to find other characteristics of a Markov chain. Consider, for instance, the finite random walk, between points  $-w$  and  $w$ , starting at  $k$ , with  $0 < k < w$ . We assume the end points are absorbing, so we may think of this as a gambler's wealth, where he is equally likely to win, lose, or draw in each period, until he is bankrupt or has reached wealth  $w$ . The recursion equations for the mean time to absorption into state  $-w$  or  $w$  are then given by

$$\begin{aligned} m_{-w} &= 0 \\ m_w &= 0 \\ m_n &= m_n/3 + m_{n-1}/3 + m_{n+1}/3 + 1 \quad -w < n < w. \end{aligned}$$

We can rewrite the recursion equation as

$$m_{n+1} = 2m_n - m_{n-1} - 3.$$

We can solve this, using the techniques of section 13.6. The associated characteristic equation is  $x^2 = 2x-1$ , with double root  $x = 1$ , so  $m_n = a + nb$ . To deal with the inhomogeneous part ( $-3$ ), we try adding a quadratic term, so  $m_n = a + bn + cn^2$ . We then have

$$a + b(n+1) + c(n^2 + 2n + 1) = 2(a + bn + cn^2) - (a + b(n-1) + c(n-1)^2) - 3$$

which simplifies to  $c = 2/3$ . To solve for  $a$  and  $b$ , we use the boundary conditions  $m_{-w} = m_w = 0$ , getting

$$m_n = \frac{3}{2}(w^2 - n^2).$$

We can use similar equations to calculate the probability  $p_n$  of being absorbed at  $-w$  if one starts at  $n$ . In this case, we have

$$\begin{aligned} p_{-w} &= 1 \\ p_w &= 0 \\ p_n &= p_n/3 + p_{n-1}/3 + p_{n+1}/3 \quad 0 < n < w. \end{aligned}$$

We now have  $p_i = a + bi$  for constants  $a$  and  $b$ . Now,  $p_{-w} = 1$  means  $a - bw = 1$ , and  $p_w = 0$  means  $a + bw = 0$ , so

$$p_i = \frac{1}{2} \left( 1 - \frac{i}{w} \right).$$

Note that the random walk is “fair” in the sense that the expecting payoff if you start with wealth  $i$  is equal to  $w(1 - p_i) - wp_i = i$ .

For an example of a denumerable Markov chain, suppose an animal is in state  $d_k = k + 1$  if it has a  $k + 1$ -day supply of food. The animal forages for food only when  $k = 0$ , and then he finds a  $k + 1$ -day supply of food with probability  $f_k$ , for  $k = 0, 1, \dots$ . This means that the animal surely finds enough food to subsist for at least one day. This is a Markov chain with  $p_{0k} = f_k$  for all  $k$ , and  $p_{k,k-1} = 1$  for  $k \geq 1$ , all other transition probabilities being zero. The recursion equations in this case are

$$u_i = u_0 f_i + u_{i+1}$$

for  $i \geq 0$ . If we let  $r_k = f_k + f_{k+1} + \dots$  for  $k \geq 0$  (so  $r_k$  is the probability of finding at least a  $k + 1$  days' supply of food when foraging), it is easy to see that  $u_k = r_k u_0$  satisfies the recursion equations; that is,

$$r_i u_0 = u_0 f_i + r_{i+1} u_0.$$

The requirement that  $\sum_i u_i = 1$  becomes  $u_0 = 1/\mu$ , where  $\mu = \sum_{k=0}^{\infty} r_k$ . To see that  $\mu$  is the expected value of the random variable  $d$ , note that

$$\begin{aligned} \mathbf{E}d &= 1f_0 + 2f_1 + 3f_2 + 4f_3 + 5f_4 + \dots \\ &= r_0 + f_1 + 2f_2 + 3f_3 + 4f_4 \dots \\ &= r_0 + r_1 + f_2 + 2f_3 + 3f_4 + \dots \\ &= r_0 + r_1 + r_2 + f_3 + 2f_4 + \dots \\ &= r_0 + r_1 + r_2 + r_3 + f_4 + \dots, \end{aligned}$$

and so on.<sup>2</sup>

We conclude that if this expected value does not exist, then no stationary distribution exists. Otherwise, the stationary distribution is given by

$$u_i = r_i/\mu \quad \text{for } i = 0, 1, \dots$$

Note that  $\mu = 1/u_0$  is the expected number of periods between visits to state 0, because  $\mu$  is the expected value of  $d$ . We can also show that  $1/u_k = \mu/r_k$  is the expected number of periods  $\mu_k$  between visits to state  $k$ , for any  $k \geq 0$ . Indeed, the fact that  $u_k = 1/\mu_k$ , where  $u_k$  is the probability of being in state  $k$  in the long run and  $\mu_k$  is the expected number of periods between visits to state  $k$ , is a general feature of Markov chains with stationary distributions. It is called the *renewal equation*.

Let us prove that  $\mu_k = \mu/r_k$  for  $k = 2$  in the preceding model, leaving the general case to the reader. From state 2 the Markov chain moves to state 0 in two periods, then requires some number  $j$  of periods before it moves to some state  $k \geq 2$ , and then in  $k - 2$  transitions moves to state 2. Thus, if we let  $v$  be the expected value of  $j$  and we let  $w$  represent the expected value of  $k$ , we have  $\mu_k = 2 + v + w - 2 = v + w$ . Now  $v$  satisfies the recursion equation

$$v = f_0(1 + v) + f_1(2 + v) + r_2(1),$$

because after a single move the system remains in state 0 with probability  $f_0$  and the expected number of periods before hitting  $k > 1$  is  $1 + v$  (the first term), or it moves to state 1 with probability  $f_1$  and the expected number of periods before hitting  $k > 1$  is  $2 + v$  (the second term), or hits  $k > 1$  immediately with probability  $r_2$  (the final term). Solving, we find that  $v = (1 + f_1)/r_2$ . To find  $w$ , note that the probability of being in state  $k$  conditional on  $k \geq 2$  is  $f_k/r_2$ . Thus  $v + w = \mu/r_2$  follows from

$$\begin{aligned} w &= \frac{1}{r_2}(2f_2 + 3f_3 + \dots) \\ &= \frac{1}{r_2}(\mu - 1 - f_1). \end{aligned}$$

<sup>2</sup>More generally, noting that  $r_k = P[d \geq k]$ , suppose  $x$  is a random variable on  $[0, \infty)$  with density  $f(x)$  and distribution  $F(x)$ . If  $x$  has finite expected value, then using integration by parts, we have  $\int_0^\infty [1 - F(x)]dx = \int_0^\infty \int_x^\infty f(y)dydx = xf(x)|_0^\infty + \int_0^\infty xf(x)dx = E[x]$ .

### 13.2 The Ergodic Theorem for Markov Chains

When are equations (13.1)-(13.3) true, and what exactly do they say? To answer this, we will need a few more concepts. Throughout, we let  $M$  be a finite or denumerable Markov chain with transition probabilities  $\{p_{ij}\}$ . We say a state  $j$  can be *reached* from a state  $i$  if  $p_{ij}^{(m)} > 0$  for some positive integer  $m$ . We say a pair of states  $i$  and  $j$  *communicates* if each is reached from the other. We say a Markov chain is *irreducible* if every pair of states communicates.

If  $M$  is irreducible, and if a stationary distribution  $u$  exists, then all the  $u_i$  in (13.1) are *strictly positive*. To see this, suppose some  $u_j = 0$ . Then by (13.2), if  $p_{ij} > 0$ , then  $p_i = 0$ . Thus, any state that reaches  $j$  in one period must also have weight zero in  $u$ . But a state  $i'$  that reaches  $j$  in two periods must pass through a state  $i$  that reaches  $j$  in one period, and because  $u_i = 0$ , we also must have  $u_{i'} = 0$ . Extending this argument, we say that any state  $i$  that reaches  $j$  must have  $u_i = 0$ , and because  $M$  is irreducible, all the  $u_i = 0$ , which violates (13.3).

Let  $q_i$  be the probability that, starting from state  $i$ , the system returns to state  $i$  in some future period. If  $q_i < 1$ , then it is clear that with probability one, state  $i$  can only occur a finite number of times. Thus, in the long run we must have  $u_i = 0$ , which is impossible for a stationary distribution. Thus in order for a stationary distribution to exist, we must have  $q_i = 1$ . We say a state  $i$  is *persistent* or *recurrent* if  $q_i = 1$ . Otherwise, we say state  $i$  is *transient*. If all the states of  $M$  are recurrent, we say that  $M$  is recurrent.

Let  $\mu_i$  be the expected number of states before the Markov chain returns to state  $i$ . Clearly, if  $i$  is transient, then  $\mu_i = \infty$ , but even if  $i$  is persistent, there is no guarantee that  $\mu_i < \infty$ . We call  $\mu_i$  the *mean recurrence time* of state  $i$ . If the mean recurrence time of state  $i$  is  $\mu_i$ ,  $M$  is in state  $i$  on average one period out of every  $\mu_i$ , so we should have  $u_i = 1/\mu_i$ . In fact, this can be shown to be true whenever the Markov chain has a stationary distribution. This is called the *renewal theorem* for Markov chains. We treat the renewal theorem as part of the ergodic theorem. Thus, if  $M$  is irreducible, it can have a stationary distribution only if  $\mu_i$  is finite, so  $u_i = 1/\mu_i > 0$ . We say a recurrent state  $i$  in a Markov chain is *null* if  $\mu_i = \infty$ , and otherwise we call the state *non-null*. An irreducible Markov chain cannot have a stationary distribution unless all its recurrent states are non-null.

We say state  $i$  in a Markov chain is *periodic* if there is some integer  $k > 1$  such that  $p_{ii}^{(k)} > 0$  and  $p_{ii}^{(m)} > 0$  implies  $m$  is a multiple of  $k$ . Otherwise, we say  $M$  is *aperiodic*. It is clear that if  $M$  has a non-null, recurrent, periodic state  $i$ , then  $M$  cannot have a stationary distribution, because we must have  $u_i = \lim_{k \rightarrow \infty} p_{ii}^{(k)} > 0$ , which is impossible unless  $p_{ii}^{(k)}$  is bounded away from zero for sufficiently large  $k$ .

An irreducible, non-null recurrent, aperiodic Markov chain is called *ergodic*. We have shown that if an irreducible Markov chain is not ergodic, it cannot have a stationary distribution. Conversely, we have the following *ergodic theorem* for Markov chains, the proof of which can be found in Feller (1950).

**THEOREM 13.1** *An ergodic Markov chain  $M$  has a unique stationary distribution, and the recursion equations (13.1)-(13.3) hold with all  $u_i > 0$ . Moreover  $u_j = 1/\mu_j$  for each state  $j$ , where  $\mu_j$  is the mean recurrence time for state  $j$ .*

We say a subset  $M'$  of states of  $M$  is *isolated* if no state in  $M'$  reaches a state not in  $M'$ . Clearly an isolated set of states is a Markov chain. We say  $M'$  is an *irreducible set* if  $M'$  is isolated and all pairs of states in  $M'$  communicate. Clearly, an irreducible set is an irreducible Markov chain. Suppose a Markov chain  $M$  consists of an irreducible set  $M'$  plus a set  $A$  of states, each of which reaches  $M'$ . Then, if  $u'$  is a stationary distribution of  $M'$ , there is a stationary distribution  $u$  for  $M$  such that  $u_i = u'_i$  for  $i \in M'$  and  $u_i = 0$  for  $i \in A$ . We can summarize this by saying that a Markov chain that consists of an irreducible set of states plus a set of transient states has a unique stationary distribution in which the frequency of the transient states is zero and the frequency of recurrent states is strictly positive. We call such a Markov chain *nearly irreducible*, with transient states  $A$  and an absorbing set of states  $M'$ .

More generally, the states of a Markov chain  $M$  can be uniquely partitioned into subsets  $A, M_1, M_2 \dots$  such that for each  $i$ ,  $M_i$  is nearly irreducible and each state in  $A$  reaches  $M_i$  for some  $i$ . The states in  $A$  are thus transient, and if each  $M_i$  is non-null and aperiodic, it has a unique stationary distribution. However,  $M$  does not have a stationary distribution unless it is nearly irreducible.

### 13.3 The Infinite Random Walk

The random walk on the line starts at zero and then, with equal probability in each succeeding period, does not move, or moves up or down one unit. It is intuitively clear that in the long run, when the system has “forgotten” its starting point, is equally likely to be in any state. Because there are an infinite number of states, the probability of being in any particular state in the long run is thus zero. Clearly this Markov chain is irreducible and aperiodic. It can be shown to be recurrent, so by the ergodic theorem, it must be null-recurrent. This means that even though the Markov random walk returns to any state with probability one, its mean recurrence time is infinite.

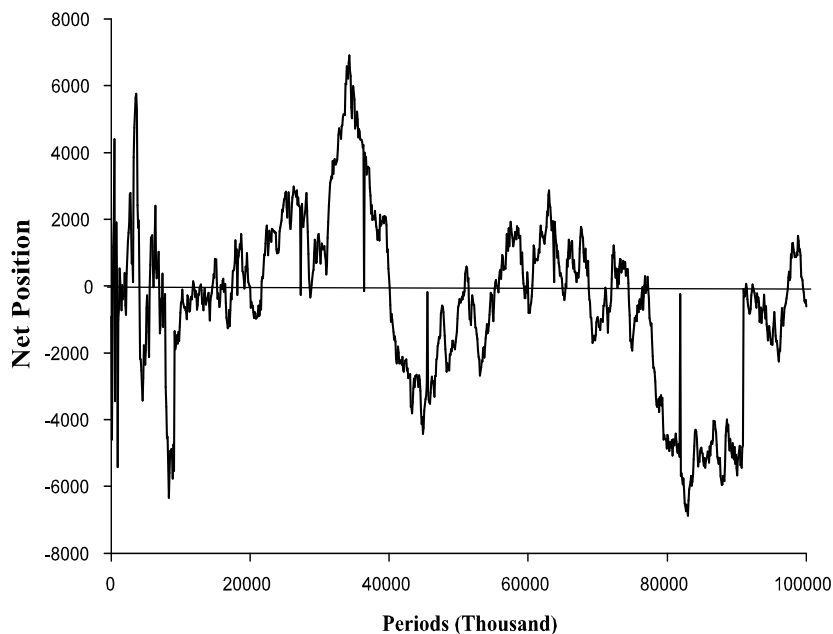


Figure 13.1. The random walk on the line

Perhaps the fact that the recurrence time for the random walk is infinite explains why individuals tend to see statistical patterns in random data that are not really there. Figure 13.1 plots the random walk for 100 million periods. The result looks biased in favor of forward from about period 20

million to 50 million, backward 75 million, forward 90 million, and forward thereafter. Of course the maximum deviation from the mean (zero) is less than 2% of the total number of periods.

### 13.4 The Sisyphian Markov Chain

As an exercise, consider the following *Sisyphian Markov chain*, in which Albert has a piano on his back and must climb up an infinite number of steps  $k = 1, 2, \dots$ . At step  $k$ , with probability  $b_k$ , he stumbles and falls all the way back to the first step, and with probability  $1 - b_k$  he proceeds to the next step. This gives the probability transition matrix

$$P = \begin{bmatrix} b_1 & 1 - b_1 & 0 & 0 & 0 & \dots \\ b_2 & 0 & 1 - b_2 & 0 & 0 & \dots \\ b_3 & 0 & 0 & 1 - b_3 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

The recursion equations for this system are

$$\begin{aligned} u_1 &= \sum u_i b_i \\ u_{k+1} &= u_k(1 - b_k) \quad \text{for } k \geq 1, \end{aligned}$$

which are satisfied only if

$$u_1(b_1 + (1 - b_1)b_2 + (1 - b_1)(1 - b_2)b_3 + \dots) = u_1,$$

so either

$$b_1 + (1 - b_1)b_2 + (1 - b_1)(1 - b_2)b_3 + \dots = 1, \quad (13.4)$$

or  $u_1 = \infty$  (note that  $u_1 \neq 0$ ). If  $b_i = \alpha$  for some  $\alpha \in [0, 1]$  and all  $i = 1, 2, \dots$ , it is easy to see that (13.4) is true (let the left-hand side equal  $x < \infty$ , subtract  $b_1$  from both sides, and divide by  $1 - b_1$ ; now the left-hand side is just  $x$  again; solve for  $x$ ).

Now, because  $\sum_i u_i = 1$ ,  $u_1$ , which must satisfy

$$u_1[1 + (1 - b_1) + (1 - b_1)(1 - b_2) + \dots] = 1.$$

This implies that the Markov chain is ergodic if  $b_k = \alpha$  for  $\alpha \in (0, 1)$  and indeed  $u_i = 1/\alpha$  for  $i = 1, \dots$ . The Markov chain is not ergodic if  $b_k = 1/k$ , however, because the mean time between passages to state 1 is infinite ( $b_1 + b_2 + \dots = \infty$ ).

### 13.5 Andrei Andreyevich's Two-Urn Problem

After Andrei Andreyevich Markov discovered the chains that bear his name, he proved the ergodic theorem for finite chains. Then he looked around for an interesting problem to solve. Here is what he came up with—this problem had been solved before, but not rigorously.

Suppose there are two urns, one black and one white, each containing  $m$  balls. Of the  $2m$  balls,  $r$  are red and the others are blue. At each time period  $t = 1, \dots$  two balls are drawn randomly, one from each urn, and each ball is placed in the other urn. Let state  $i$  represent the event that there are  $i \in [0, \dots, r]$  red balls in the black urn. What is the probability  $u_i$  of state  $i$  in the long run?

Let  $P = \{p_{ij}\}$  be the  $(r + 1) \times (r + 1)$  probability transition matrix. To move from  $i$  to  $i - 1$ , a red ball must be drawn from the black urn, and a blue ball must be drawn from the white urn. This means  $p_{i,i-1} = i(m - r + i)/m^2$ . To remain in state  $i$ , either both balls drawn are red or both are blue,  $p_{i,i} = (i(r - i) + (m - i)(m - r + i))/m^2$ . To move from  $i$  to  $i + 1$ , a blue ball must be drawn from the black urn, and a red ball must be drawn from the white urn. This means  $p_{i,i+1} = (m - i)(r - i)/m^2$ . All other transition probabilities are zero.

The recursion equations in this case are given by

$$u_i = u_{i-1}p_{i-1,i} + u_i p_{ii} + u_{i+1}p_{i+1,i} \quad (13.5)$$

for  $i = 0, \dots, r + 1$ , where we set  $u_{-1} = u_{r+2} = 0$ . I do not know how Andrei solved these equations, but I suspect he guessed at the answer and then showed that it works. At any rate, that is what I shall do. Our intuition concerning the ergodic theorem suggests that in the long run the probability distribution of red balls in the black urn are the same as if  $m$  balls were randomly picked from a pile of  $2m$  balls (of which  $r$  are red) and put in the black urn. If we write the number of combinations of  $n$  things taken  $r$  at a time as  $\binom{n}{r} = n!/r!(n - r)!$ , then  $u$  should satisfy

$$u_i = \binom{m}{i} \binom{m}{r-i} / \binom{2m}{r}.$$

The denominator in this expression is the number of ways the  $r$  red balls can be allocated to the  $2m$  possible positions in the two urns, and the numerator is the number of ways this can be done when  $i$  red balls are in the black urn. You can check that  $u$  now satisfies the recursion equations.

### 13.6 Solving Linear Recursion Equations

In analyzing the stationary distribution of a Markov chain, we commonly encounter an equation of the form

$$u_n = a_1 u_{n-1} + a_2 u_{n-2} + \dots + a_k u_{n-k}, \quad (13.6)$$

along with some boundary conditions, including  $u_i \geq 0$  for all  $i$  and  $\sum_i u_i = 1$ . Note that this recursion equation is *linear* in the sense that if  $u_n = g_i(n)$  for  $i = 1, \dots, m$  are  $m$  solutions, then so are all the weighted sums of the form  $u_n = \sum_{j=1}^m b_j g(j)$  for arbitrary weights  $b_1, \dots, b_m$ .

A general approach to solving such equations is presented by Elaydi (1999) in the general context of difference equations. We here present a short introduction to the subject, especially suited to analyzing Markov chains. First, form the associated  $k$ -degree *characteristic equation*

$$x^n = a_1 x^{n-1} + a_2 x^{n-2} + \dots + a_n, \quad (13.7)$$

The general solution to (13.6) is the weighted sum, with arbitrary coefficients, of solutions  $g(n)$  of the following form. First, suppose  $r$  is a non-repeated root of (13.7). Then  $g(n) = r^n$  is a solution to (13.6). If the root  $r$  of (13.7) is repeated  $m$  times, then  $g(n) = n^j r^n$  are independent solutions for  $j = 1, \dots, m$ . Now, choose the weights of the various terms to satisfy the system's boundary conditions.

For instance, the stationary distribution for the reflecting boundaries model with  $n$  states, given by (13.4), satisfies

$$u_k = 2u_{k-1} - u_{k-2},$$

for  $k = 2, \dots, n-1$ , with boundary conditions  $u_1 = u_2/2 = u_{n-1}/2 = u_n$  and  $u_1 + \dots + u_n = 1$ . The characteristic equation is  $x^2 = 2x - 1$ , which has the double root  $x = 1$ . Thus the general form of the solution is  $u_k = a \cdot 1^k + bk \cdot 1^k = a + bk$ . The symmetry condition then implies that  $b = 0$ , and the condition  $\sum_i u_i = 1$  implies  $a = 1/(n-1)$ .

Sometimes the recursion equations have an *inhomogeneous* part, as the  $g(i)$  in

$$u_i = u_{i-1} p_{i-1,i} + u_i p_{ii} + u_{i+1} p_{i+1,i} + g(i) \quad (13.8)$$

There is no general rule for finding the solution to the inhomogeneous part, but generally trying low-degree polynomials works.

### 13.7 Good Vibrations

Consider the pure coordination game in the diagram. We can check using the techniques of chapter 6 that there are two pure-strategy equilibria,  $ll$  and  $rr$ , as well as a mixed strategy equilibrium. If we represent the out-of-equilibrium dynamics of the game using

	$l$	$r$
$l$	5,5	0,0
$r$	0,0	3,3

a replicator process (see chapter 12), the pure strategy equilibria will be stable and the mixed strategy equilibrium unstable. But the concept of stability that is used, although at first glance compelling and intuitive, may be unrealistic in some cases. The idea is that if we start at the equilibrium  $ll$ , and we subject the system to a small disequilibrium shock, the system will move back into equilibrium. But in the real world, dynamical systems may be *constantly* subject to shocks, and if the shocks come frequently enough, the system will not have time to move back close to equilibrium before the next shock comes.

The evolutionary models considered in chapters 10 and 12 are certainly subject to continual random “shocks,” because agents are paired randomly, play mixed strategies with stochastic outcomes, and update their strategies by sampling the population. We avoided considering the stochastic nature of these processes by implicitly assuming that random variables can be replaced by their expected values, and mutations occur infrequently compared with the time to restore equilibrium. But these assumptions need not be appropriate.

We may move to stochastic differential equations, where we add a random error term to the right-hand side of an equation such as (11.1). This approach is very powerful, but uses sophisticated mathematical techniques, including stochastic processes and partial differential equations.<sup>3</sup> Moreover, applications have been confined mainly to financial economics. Applying the approach to game theory is very difficult, because stochastic differential equations with more than one independent variable virtually never have a closed-form solution. Consider the following alternative approach, based on the work of H. Peyton Young (1998) and others. We start by modeling adaptive learning with and without errors.

<sup>3</sup>For relatively accessible expositions, see Dixit 1993 and Karlin and Taylor 1981.

### 13.8 Adaptive Learning

How does an agent decide what strategy to follow in a game? We have described three distinct methods so far in our study of game theory. The first is to determine the expected behavior of the other players and choose a best response (“rational expectations”). The second is to inherit a strategy (e.g., from one’s parents) and blindly play it. The third is to mimic another player by switching to the other player’s strategy, if it seems to be doing better than one’s own. But there is a fourth, and very commonly followed, *modus operandi*: follow the history of how other players have played against you in the past, and choose a strategy for the future that is a best response to the past play of others. We call this *adaptive learning*, or *adaptive expectations*.

To formalize this, consider an evolutionary game in which each player has limited memory, remembering only  $h = \{h_1, h_2, \dots, h_m\}$ , the last  $m$  moves of the players with whom he has been paired. If the player chooses the next move as a best response to  $h$ , we say the player follows adaptive learning.

Suppose, for instance, two agents play the coordination game in section 13.7, but the payoffs to  $ll$  and  $rr$  are both 5, 5. Let  $m = 2$ , so the players look at the last two actions chosen by their opponents. The best response to  $ll$  is thus  $l$ , the best response to  $rr$  is  $r$ , and the best response to  $rl$  or  $lr$  is any combination of  $l$  and  $r$ . We take this combination to be: play  $l$  with probability 1/2 and  $r$  with probability 1/2. There are 16 distinct “states” of the game, which we label  $abcd$ , where each of the letters can be  $l$  or  $r$ ,  $b$  is the previous move by player 1,  $a$  is player 1’s move previous to this,  $d$  is the previous move by player 2, and  $c$  is player 2’s move previous to this. For instance,  $llrl$  means player 1 moved  $l$  on the previous two rounds, whereas player 2 moved first  $r$  and then  $l$ .

We can reduce the number of states to 10 by recognizing that because we do not care about the order in which the players are counted, a state  $abcd$  and a state  $cdab$  are equivalent. Eliminating redundant states, and ordering the remaining states alphabetically, the states become  $llll$ ,  $lllr$ ,  $llrl$ ,  $llrr$ ,  $lrlr$ ,  $lrrl$ ,  $lrrr$ ,  $rlrl$ ,  $rlrr$ , and  $rrrr$ . Given any state, we can now compute the probability of a transition to any other state on the next play of the game. For instance,  $llll$  (and similarly  $rrrr$ ) is an *absorbing* state in the sense that, once it is entered, it stays there forever. The state  $lllr$  goes to states  $llrl$  and  $lrrl$ , each with probability 1/2. The state  $llrl$  goes either to  $llll$  where it stays forever, or to  $lllr$ , each with probability

1/2. The state  $lr lr$  goes to  $rlrl$  and  $rrrr$  each with probability 1/4, and to  $rlrr$  with probability 1/2. And so on.

We can summarize the transitions from state to state in a  $10 \times 10$  matrix  $M = (m_{ij})$ , where  $m_{abcd,efgi}$  is the probability of moving from state  $abcd$  to state  $efgi$ . We call  $M$  a *probability transition matrix*, and the dynamic process of moving from state to state is a *Markov chain* (§13.1). Because matrices are easier to describe and manipulate if their rows and columns are numbered, we will assign numbers to the various states, as follows:  $llll = 1, ll lr = 2, \dots, rrrr = 10$ . This gives us the following probability transition matrix:

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.25 & 0.5 & 0.25 \\ 0 & 0 & 0.25 & 0.25 & 0 & 0.25 & 0.25 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0.5 \\ 0.25 & 0.5 & 0 & 0 & 0.25 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.5 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Also, if we represent the 10 states by the 10 10-dimensional row vectors  $\{v_1, \dots, v_{10}\}$ , where  $v_1 = (1, 0, \dots, 0)$ ,  $v_2 = (0, 1, 0, \dots, 0)$ , and so on, then it is easy to see that, if we are in state  $v_i$  in one period, the probability distribution of states in the next period is just  $v_i M$ , meaning the product of  $v_i$ , which is a  $1 \times 10$  row vector, and  $M$ , which is a  $10 \times 10$  matrix, so the product is another  $1 \times 10$  row vector. It is also easy to see that the sum of the entries in  $v_i M$  is unity and that each entry represents the probability that the corresponding state will be entered in the next period.

If the system starts in state  $i$  at  $t = 0$ ,  $v_i M$  is the probability distribution of the state it is in at  $t = 1$ . The probability distribution of the state the system at  $t = 2$  can be written as

$$v_i M = p_1 v_1 + \dots + p_{10} v_{10}.$$

Then, with probability  $p_j$  the system has probability distribution  $v_j M$  in the second period, so the probability distribution of states in the second period is

$$p_1 v_1 M + \dots + p_{10} v_{10} M = v_i M^2.$$

Similar reasoning shows that the probability distribution of states after  $k$  periods is simply  $v_i M^k$ . Thus, just as  $M$  is the probability transition matrix for one period, so is  $M^k$  the probability transition matrix for  $k$  periods. To find out the long-run behavior of the system, we therefore want to calculate

$$M^* = \lim_{k \rightarrow \infty} M^k.$$

I let Mathematica, the computer algebra software package, calculate  $M^k$  for larger and larger  $k$  until the entries in the matrix stopped changing or became vanishingly small, and I came up with the following matrix:

$$M^* = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2/3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/3 \\ 5/6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/6 \\ 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 \\ 1/3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2/3 \\ 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 \\ 1/6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5/6 \\ 2/3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/3 \\ 1/3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

In other words, no matter where you start, you end up in one of the absorbing states, which is a Pareto-optimal Nash equilibrium. We call pure-strategy Nash equilibria in which all players choose the same strategy *conventions* (Young 1998). We conclude that *adaptive learning leads with probability 1 to a convention*.

### 13.9 The Steady State of a Markov Chain

There is a simpler way to compute  $M^*$  in the previous case. The computation also gives a better intuitive feel for the steady-state solution to the adaptive learning dynamical system generated by a pure coordination game. We know that whatever state we start the system in, we will end up in either state  $llll$  or state  $rrrr$ . For state  $abcd$ , let  $P[abcd]$  be the probability that we end up in  $llll$  starting from  $abcd$ . Clearly,  $P[llll] = 1$  and  $P[rrrr] = 0$ . Moreover,  $P[lllr] = P[llrl]/2 + P[lrrl]/2$ , because  $lllr$  moves to either  $llrl$  or to  $lrrl$  with equal probability. Generalizing, you

can check that, if we define

$$v = (P[llll], P[lllr], \dots, P[rrrr])',$$

the column vector of probabilities of being absorbed in state  $llll$ , then we have

$$Mv = v.$$

If we solve this equation for  $v$ , subject to  $v[1] = 1$ , we get

$$v = (1, 2/3, 5/6, 1/2, 1/3, 1/2, 1/6, 2/3, 1/3, 0)',$$

which then must be the first column of  $M^*$ . The rest of the columns are zero, except for the last, which must have entries so the rows each sum up to unity. By the way, I would not try to solve the equation  $Mv = v$  by hand unless you're a masochist. I let Mathematica do it ( $v$  is a *left eigenvector* of  $M$ , so Mathematica has a special routine for finding  $v$  easily).

### 13.10 Adaptive Learning II

Now consider the pure coordination game illustrated in section 13.7, where the  $ll$  convention Pareto-dominates the  $rr$  convention. How does adaptive learning work in such an environment? We again assume each player finds a best response to the history of the other player's previous two moves. The best response to  $ll$  and  $rr$  are still  $l$  and  $r$ , respectively, but now the best response to  $rl$  or  $lr$  is also  $l$ . Now, for instance,  $lllr$  and  $lr lr$  both go to  $llll$  with probability 1. The probability transition matrix now becomes as shown.

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

To calculate

$$M^* = \lim_{k \rightarrow \infty} M^k$$

is relatively simple, because in this case  $M^k = M^4$  for  $k \geq 4$ . Thus, we have

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

In other words, if you start in state  $rrrr$ , you stay there; otherwise, after four steps you arrive at  $llll$  and remain there forever. We conclude that *with adaptive learning, if the system starts in a nonconventional state, it always ends up in the Pareto-efficient conventional state.*

### 13.11 Adaptive Learning with Errors

We now investigate the effect on a dynamic adaptive learning system when players are subject to error. Consider the pure coordination game illustrated in section 13.7, but where the payoffs to  $ll$  and  $rr$  are equal. Suppose each player finds a best response to the history of the other player's previous two moves with probability  $1 - \epsilon$ , but chooses incorrectly with probability  $\epsilon > 0$ . The probability transition matrix now becomes

$$M = \begin{pmatrix} a & 2b & 0 & 0 & e & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & c & d & 0 & c & d & 0 & 0 & 0 \\ c & 1/2 & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b & e & 0 & a & b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a & 0 & 1/4 & 1/2 & 1/4 \\ 0 & 0 & 1/4 & 1/4 & 0 & 1/4 & 1/4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & 1/2 & c \\ 1/4 & 1/2 & 0 & 0 & 1/4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & d & d & 0 & c & c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & e & 2b & a \end{pmatrix},$$

where  $a = (1-\epsilon)^2$ ,  $b = \epsilon(1-\epsilon)$ ,  $c = (1-\epsilon)/2$ ,  $d = \epsilon/2$ , and  $e = \epsilon^2$ . Note that now *there are no absorbing states*. To see what happens in the long run,

suppose  $\epsilon = 0.01$ , so errors occur 1% of the time. Using Mathematica to calculate  $M^*$ , we find *all the rows are the same*, and each row has the entries

$$(0.442 \ 0.018 \ 0.018 \ 0.001 \ 0.0002 \ 0.035 \ 0.018 \ 0.0002 \ 0.018 \ 0.442)$$

In other words, you spend about 88.4% of the time in one of the conventional states, and about 11.6% of the time in the other states.

It should be intuitively obvious how the system behaves. If the system is in a conventional state, say  $llll$ , it remains there in the next period with probability  $(1 - \epsilon)^2 = 98\%$ . If one player makes an error, the state moves to  $lllr$ . If there are no more errors for a while, we know it will return to  $llll$  eventually. Thus, it requires multiple errors to “kick” the system to a new convention. For instance,  $llll \rightarrow lllr \rightarrow lrrr \rightarrow rrrr$  can occur with just two errors:  $llll \rightarrow lllr$  with one error,  $lllr \rightarrow lrrr$  with one error, and  $lrrr \rightarrow rrrr$  with no errors, but probability  $1/2$ . We thus expect convention flips about every 200 plays of the game.

To test our “informed intuition,” I ran 1000 repetitions of this stochastic dynamical system using Mathematica. Figure 13.2 reports on the result.

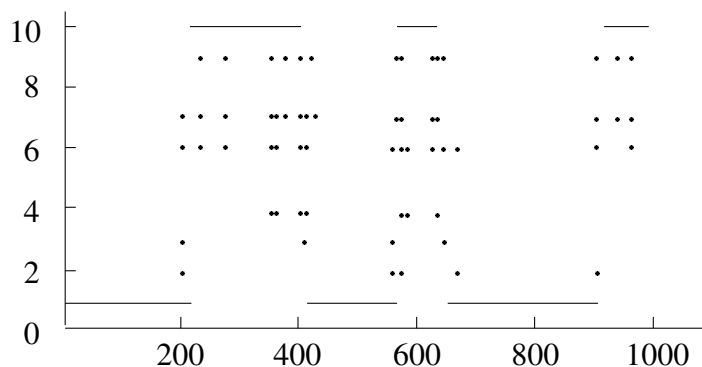


Figure 13.2. An agent-based model adaptive learning with errors.

### 13.12 Stochastic Stability

We define a state in a stochastic dynamical system to be *stochastically stable* if the long-run probability of being in that state does not become zero or vanishingly small as the rate of error  $\epsilon$  goes to zero. Clearly, in the previous example  $llll$  and  $rrrr$  are both stochastically stable and no other state is. Consider the game in section 13.7. It would be nice if the Pareto-dominant

equilibrium  $ll$  were stochastically stable, and no other state were stochastically stable. We shall see that is the case. Now the probability transition matrix becomes

$$M = \begin{pmatrix} a & 2b & 0 & 0 & e & 0 & 0 & 0 & 0 & 0 \\ 0 & 2b & a & 0 & e & 0 & 0 & 0 & 0 & 0 \\ a & 2b & 0 & 0 & e & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b & e & 0 & a & b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & b & a & b \\ 0 & 0 & a & b & 0 & b & e & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & b & a & b \\ a & 2b & 0 & 0 & e & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b & e & 0 & a & b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & e & 2b & a \end{pmatrix},$$

where  $a = (1-\epsilon)^2$ ,  $b = \epsilon(1-\epsilon)$ , and  $e = \epsilon^2$ . Again there are no absorbing states. If  $\epsilon = 0.01$ , we calculate  $M^*$ , again we find *all the rows are the same*, and each row has the entries

$$(0.9605 \quad 0.0198 \quad 0.0198 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0).$$

In other words, the system spends 96% of the time in the Pareto-dominant conventional states and virtually all of the remaining time in “nearby states.” It is clear (though it should be formally proved) that  $ll$  is the only stochastically stable state.