6

Mixed-strategy Nash Equilibria

Leges sine moribus vanae

Horace

This chapter presents a variety of games with mixed-strategy Nash equilibria, many in the form of problems to be solved by the reader. Some mixed-strategy equilibria, such as throwing fingers (§3.8), are intuitively obvious. Others, such as the hawk-dove equilibrium (§3.10) are not intuitive, but the equilibrium depends in plausible ways on the parameters of the problem. For instance, as the cost of injury w increases in the hawk-dove game, the probability of playing hawk declines, and as the value of the territory v increases, the probability of playing hawk also increases. However, the mixed-strategy equilibrium in battle of the sexes (§3.9), is implausible because it suggest that a player's Nash strategy does not depend on the relative strength of preferences for the two pure-strategy equilibrium outcomes. Indeed, you will ascertain later that this mixed-strategy equilibrium is not stable in an evolutionary dynamic (§12.17).

6.1 The Algebra of Mixed Strategies

There is a simple way to do the algebra of mixed strategies. Examples in this case are worth more than formalities, so I will give one. The reader will find it easy to generalize.

Suppose Alice has strategy set $\{L, R\}$ and uses mixed strategy $\sigma = \alpha L + (1-\alpha)R$, whereas Bob has strategy set $\{U, D\}$ and uses mixed strategy $\tau = \beta U + (1-\beta)D$. We can then think of the payoff to Alice, $\pi_1(\sigma, \tau)$, as the value to Alice of the compound lottery in figure 6.1.

We can reduce this compound lottery to a simple lottery with four payoffs, (L, U), (L, D), (R, U), and (R, D), with probabilities $\alpha\beta$, $\alpha(1-\beta)$, $(1-\alpha)\beta$, and $(1-\alpha)(1-\beta)$, respectively. The payoff to this lottery for player i is then

$$\pi_i(\sigma, \tau) = \alpha\beta\pi_i(L, U) + \alpha(1-\beta)\pi_i(L, D) \\ + (1-\alpha)\beta\pi_i(R, U) + (1-\alpha)(1-\beta)\pi_i(R, D).$$

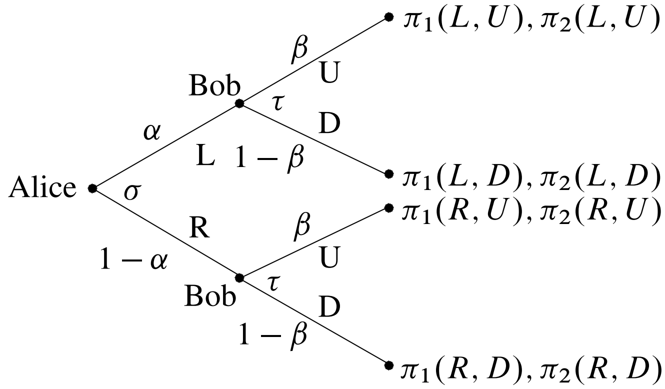

Figure 6.1. Mixed strategies as compound lotteries

Another way to think of this is to define the *product* of mixed strategies by

$$\begin{aligned}\sigma\tau &= (\alpha L + (1-\alpha)R)(\beta U + (1-\beta)D)\\ &= \alpha\beta LU + \alpha(1-\beta)LD + (1-\alpha)\beta RU + (1-\alpha)(1-\beta)RD,\end{aligned}$$

and then the payoff to Alice from the strategy pair (σ, τ) is

$$\begin{aligned}\pi_1(\sigma,\tau) &= \pi_1(\alpha\beta LU + \alpha(1-\beta)LD + (1-\alpha)\beta RU + (1-\alpha)(1-\beta)RD)\\ &= \alpha\beta\pi_1(L,U) + \alpha(1-\beta)\pi_1(L,D)\\ &\qquad + (1-\alpha)\beta\pi_1(R,U) + (1-\alpha)(1-\beta)\pi_1(R,D).\end{aligned}$$

y

6.2 Lions and Antelope

	BA	LA
BA	$c_b/2$,$c_b/2$	c_b,c_l
LA	c_l,c_b	$c_l/2$,$c_l/2$

Two lions out hunting see Big Antelope and Little Antelope in the distance. They will surely catch whatever prey they chase, whether alone or together. However, if they pick different antelopes to chase, there is no need to share, whereas if they go after the same antelope, each will get only half of the kill and the other antelope will escape. Suppose their decisions are independent, the caloric value of Big Antelope is c_b, the caloric value of Little Antelope is c_l and $0 < c_l < c_b$. Let BA be the strategy "hunt Big Antelope," and let LA be the strategy "hunt Little Antelope." The normal form game is shown in the diagram. The lions cannot distinguish between player 1 and player 2, so only symmetric Nash equilibria are acceptable.

If (BA,BA) is to be a pure-strategy equilibrium, it is necessary that $c_b/2 \geq c_l$ and it is easy to see that this condition is also sufficient. Because $c_b > c_l$, it is easy to see that (LA,LA) is not Nash. To find the mixed-strategy equilibrium, we assume (BA,BA) is not Nash, so $c_b < 2c_l$. Let α be the probability a lion uses BA. Then the payoff to the other lion from using BA is

$$\alpha\frac{c_b}{2} + (1-\alpha)c_b = c_b - \alpha\frac{c_b}{2},$$

and the payoff to using LA is

$$\alpha c_l + (1-\alpha)\frac{c_l}{2} = (1+\alpha)\frac{c_l}{2}.$$

Equating these two, we get

$$\alpha = \frac{2c_b - c_l}{c_b + c_l}.$$

For both lions, the payoff to the mixed strategy is equal to the payoff to playing either one of the strategies in support of the mixed strategy, so this payoff is equal to the payoff to BA, which is

$$c_b - \alpha\frac{c_b}{2} = c_b\left(1 - \frac{\alpha}{2}\right) = c_b\frac{3c_l}{2(c_b + c_l)}. \tag{6.1}$$

It is easy to check that the preceding fraction is greater than 1/2, so they should play the mixed strategy.

One can also calculate the expected payoff using the payoff to LA instead of the payoff to BA:

$$(1+\alpha)\frac{c_l}{2} = \left(\frac{3c_b}{c_b + c_l}\right)\frac{c_l}{2} = c_l\frac{3c_b}{2(c_b + c_l)}, \tag{6.2}$$

which is the same.

6.3 A Patent Race

Rapoport and Amaldoss (1997) set up a "patent race" game in which a "weak" player is given an endowment of 4, any integral amount of which could be invested in a project with a return of 10. However a "strong" player is given an endowment of 5 and both players are instructed that whichever player invests the most will receive the return of 10 for the patent and if there is a tie, neither gets the return of 10. It is clear that the weak player has five pure strategies (invest 0, 1, 2, 3, or 4) and the strong player will choose from 0, 1, 2, 3, 4, or 5. The payoffs to the game are as in section 4.3, matrix (g).

Show that each player has three strategies remaining after the iterated elimination of dominated strategies and then show that the game has a mixed-strategy equilibrium in which each player uses the remaining strategies with probabilities (3/5,1/5,1/5) and (1/5,1/5,3/5), respectively. Show also the the expected payoff to the players is $(4, 10)$. This is in fact the unique Nash equilibrium of the game, although this is a bit harder to show.

6.4 Tennis Strategy

In tennis, the server can serve to either the receiver's backhand or the receiver's forehand. The receiver can anticipate that the ball will come to either the forehand or backhand side. A receiver who anticipates correctly is more likely to return the ball. On the other hand, the server has a stronger backhand than forehand serve. Therefore, the receiver will return a correctly anticipated backhand serve with 60% probability and a correctly anticipated forehand serve with 90% probability. A receiver who wrongly anticipates a forehand hits a good return 20% of the time, whereas a receiver who wrongly anticipates a backhand hits a good return 30% of the time. The normal for game is shown in the diagram. Find the Nash equilibria of this game.

	b_r	f_r
b_s	$0.4, 0.6$	$0.7, 0.3$
f_s	$0.8, 0.2$	$0.1, 0.9$

6.5 Preservation of Ecology Game

Each of three firms (1, 2, and 3) uses water from a lake for production purposes. Each has two pure strategies: purify sewage (strategy 1) or divert it back into the lake (strategy 2). We assume that if zero or one firm diverts its sewage into the lake, the water remains pure, but if two or more firms do, the water is impure and each firm suffers a loss of 3. The cost of purification is 1.

We will show that the Nash equilibria are: (a) One firm always pollutes and the other two always purify, (b) All firms always pollute, (c) Each firm purifies with probability $1/(3+\sqrt{3})$, (d) Each firm purifies with probability $1/(3-\sqrt{3})$, or (e) One firm always purifies and the other two purify with probability 2/3.

The pure-strategy cases (a) and (b) are obvious. For the completely mixed equilibria (c) and (d), suppose x, y and z are the probabilities the three firms purify, $x, y, z > 0$. If firm 3 purifies, its expected payoff is $-xy - x(1-y) - y(1-x) - 4(1-x)(1-y)$. If firm 3 pollutes, its payoff is $-3x(1-y) - 3(1-x)y - 3(1-x)(1-y)$. Because firm 3 uses a completely mixed strategy, these must be equal, so after simplification we have $(1-3x)(1-3y) = 3xy$. Solving and repeating for the other two firms, we get the two desired solutions. Case (e) is derived by assuming one firm choose purify with probability 1 and then finding the completely mixed strategies of the other firms.

6.6 Hard Love

A mother wants to help her unemployed son financially, but she does not want to contribute to his distress by allowing him to loaf around. Therefore, she announces that she *may* help her son in the current period if he does not find a job. The son, however, seeks work only if he cannot depend on his mother for support and may not find work even if he searches. The payoff matrix is as shown. It is clear from the diagram that there are no pure-strategy Nash equilibria. Find the unique mixed-strategy equilibrium.

Mom \ Son	Seek Work	Watch Soaps
Help Son	3,2	−1,3
Hard Love	−1,1	0,0

6.7 Advertising Game

Three firms (players 1, 2, and 3) put three items on the market and can advertise these products either on morning or evening TV. A firm advertises exactly once per day. If more than one firm advertises at the same time, their profits are 0. If exactly one firm advertises in the morning, its profit is 1 and if exactly one firm advertises in the evening, its profit is 2. Firms must make their daily advertising decisions simultaneously.

There is one set of equilibria in which one firm always chooses morning, another always chooses evening, and the third chooses morning with any probability. Moreover, these are the *only* Nash equilibria in which at least one firm uses a pure strategy. To see this, suppose first that firm 1 chooses the pure strategy M (morning). If both firms 2 and 3 choose mixed strategies, then one of them could gain by shifting to pure strategy E (evening). To see this, let the two mixed strategies be $\alpha M + (1-\alpha)E$ for firm 2 and $\beta M + (1-\beta)E$ for firm 3. Let $\pi_i(s_1 s_2 s_3)$ be the payoff to player i when the three firms use pure strategies $s_1 s_2 s_3$. Then, the payoff to M for firm 2 is

$$\begin{aligned}\pi_2 &= \alpha\beta\pi_2(MMM) + \alpha(1-\beta)\pi_2(MME) + (1-\alpha)\beta\pi_2(MEM) \\ &\qquad + (1-\alpha)(1-\beta)\pi_2(MEE) \\ &= \alpha\beta(0) + \alpha(1-\beta)(0) + (1-\alpha)\beta(2) + (1-\alpha)(1-\beta)(0) \\ &= 2(1-\alpha)\beta.\end{aligned}$$

Because $0 < \beta$ by definition, this is maximized by choosing $\alpha = 0$, so firm 2 should use pure strategy E. This contradicts our assumption that both firms 1 and 2 use mixed strategies.

A similar argument holds if firm 1 uses pure strategy E. We conclude that if firm 1 uses a pure strategy, at least one of the other two firms will use a pure strategy. The firm that does will not use the same pure strategy as firm 1, because this would not be a best response. Therefore, two firms use opposite pure strategies and it does not matter what the third firm does. Now we repeat the whole analysis assuming firm 2 uses a pure strategy, with clearly the same outcome. Then, we do it again for firm 3. This proves that if one firm uses a pure strategy, at least two firms use a pure strategy, which concludes this part of the problem.

To find the mixed-strategy equilibria, let x, y, and z be the probabilities of advertising in the morning for firms 1, 2, and 3. The expected return to 1 of advertising in the morning is $(1-y)(1-z)$ and in the evening it is $2yz$. If these are equal, any choice of x for firm 1 is Nash. But equality means $1-y-z-yz = 0$, or $y = (1-z)/(1+z)$. Now repeat for firms 2 and 3, giving the equalities $y = (1-z)/(1+z)$ and $z = (1-x)/(1+x)$. Solving simultaneously, we get $x = y = z = \sqrt{2}-1$. To see this, substitute $y = (1-z)/(1+z)$ in $x = (1-y)/(1+y)$, getting

$$x = \frac{1-y}{1+y} = \frac{1-\frac{1-z}{1+z}}{1+\frac{1-z}{1+z}} = z.$$

Thus, $x = (1-x)/(1+x)$, which is a simple quadratic equation, the only root of which between 0 and 1 is $\sqrt{2}-1$. Thus, this is Nash.

To show that there are no other Nash equilibria, suppose $0 < x < 1$ and $0 < y < 1$. We must show $0 < z < 1$, which reproduces equilibrium (b). But $0 < x < 1$ implies $(1+y)(1+z) = 2$ (*why?*) and $0 < y < 1$ implies $(1+x)(1+z) = 2$. If $z = 0$, then $x = y = 1$, which we assumed is not the case. If $z = 1$ then $x = y = 0$, which is also not the case. This proves it.

6.8 Robin Hood and Little John

Robin Hood and Little John both want to cross a rope bridge at the same time. There is only room for one. Each has two strategies: go (G) and wait (W). It takes Robin Hood and Little John times τ_r and τ_{lj}, respectively, to cross the bridge. If both go at the same time, they fight it out at cost $\delta > 0$, after which the winner crosses the bridge. The probability of winning is 1/2 for each. If both wait, they play a polite little game of Alphonse and Gaston, at a cost $\epsilon > 0$ and one of them eventually crosses first, again with probability 1/2. We assume $0 < \epsilon < \delta$, while τ_r and τ_{lj} represent the cost of waiting for the other to cross the bridge.

Write the payoff matrix for this game, considering each player's cost as not including the necessary crossing time for himself and find all of the Nash equilibria, writing α_r and α_{lj} for the probabilities of Robin Hood and Little John going. Show that the larger δ, the less likely a go-go situation emerges and find the socially optimal δ. Show that if Robin Hood always waits, he would gain by an appropriate reduction in the costs of fighting but would not gain by an increase in the costs of fighting.

6.9 The Motorist's Dilemma

Alice and Bob, traveling in opposite directions, come to an intersection and each wants to turn left, so one must wait for the other. The time one must wait while the other turns left is the same for both and is equal to $\tau > 0$. The loss if both wait is $\epsilon > 0$ each and then one of the two is randomly chosen to turn, the other incurring the additional cost τ of waiting. If both go at the same time, the loss is $\delta > \epsilon$ each and then one of the two is randomly chosen to turn, the other incurring the additional cost τ of waiting. To the two strategies G (go) and W (wait), we add a third, C (contingent). Playing C means choosing W if the other driver chooses G and choosing G if the other driver chooses W. If both drivers choose C, we treat this as a foul-up equivalent to GG. Find the Nash equilibria of the game.

6.10 Family Politics

Male	Female: Coy	Female: Loose
Faithful	$v - r - w$ $v - r - w$	$v - r$ $v - r$
Philanderer	0 0	v $v - 2r$

In certain species of bird (actually, this is true of many bird species) males are faithful or philanderers, females are coy or loose. Coy females insist on a long courtship before copulating, while loose females do not. Faithful males tolerate a long courtship and help rear their young, while philanderers do not wait and do not help. Suppose v is the value of having offspring to either a male or a female, $2r > 0$ is the total cost of rearing an offspring and $w > 0$ the cost of prolonged courtship to both male and female. We assume $v > r + w$. This means that if courtship leads to sharing the costs of raising an offspring, then it is worth it to both birds. The normal form matrix is shown in the diagram.

a. Show that if $v > 2r$, there is one Nash equilibrium with only loose females and only philandering males.
b. Show that if $v < 2r$, there is a unique completely mixed strategy for males and females. The fraction q of females who are coy is then given by $q = r/(v - w)$ and the fraction p of males who are philanderers is given by $w/(2r + w - v)$.

6.11 Frankie and Johnny

Frankie must pay Johnny a certain amount of money as compensation for shooting Johnny's lover, but they disagree on the amount. They agree on a negotiator, who will pick whichever of Frankie's bid x_f and Johnny's bid x_j is closer to the negotiator's opinion y. We assume $x_f, x_j \in [0, 1]$. Frankie and Johnny do not know y, but they know it is drawn from a distribution F with a continuous density f, such that $\Pr\{y < \tilde{y}\} = F(\tilde{y})$. Find the equilibrium values of x_f and x_j in terms of f and F. Solve explicitly in case y is drawn from a uniform distribution.

6.12 A Card Game

There are two players, each of whom bets \$1 and receives a number between 0 and 1 (uniformly distributed). Each player observes only his number. Player 1 can either fold or raise \$5. Player 2 can either fold or see. If neither folds, the player with the higher number wins.

The only undominated strategy for each player is to choose a critical level x_i^* and to fold iff $x_i < x_i^*$. Let (x_1^*, x_2^*) be Nash strategies. The payoff to player 1 is

$$
\begin{aligned}
-1 \cdot \mathrm{P}[x_1 < x_1^*] &+ 1 \cdot \mathrm{P}[x_1 > x_1^*, x_2 < x_2^*] \\
&- 6 \cdot \mathrm{P}[x_1 > x_1^*, x_2 > x_2^*, x_2 > x_1] \\
&+ 6 \cdot \mathrm{P}[x_1 > x_1^*, x_2 > x_2^*, x_2 < x_1].
\end{aligned}
$$

Clearly, we have

$$P[x_1 < x_1^*] = x_1^*, \qquad P[x_1 > x_1^*, x_2 < x_2^*] = (1 - x_1^*)x_2^*.$$

We also know

$$\begin{aligned} P[x_1 > x_1^*, x_2 > x_2^*, x_2 > x_1] + P[x_1 > x_1^*, x_2 > x_2^*, x_2 < x_1] \\ = P[x_1 > x_1^*, x_2 > x_2^*] \\ = (1 - x_1^*)(1 - x_2^*). \end{aligned}$$

To evaluate $P[x_1 > x_1^*, x_2 > x_2^*, x_2 > x_1]$, suppose $x_1^* > x_2^*$. Then,

$$P[x_1 > x_1^*, x_2 > x_2^*, x_2 > x_1] = P[x_1 > x_1^*, x_2 > x_1] = \frac{(1 - x_1^*)^2}{2}.$$

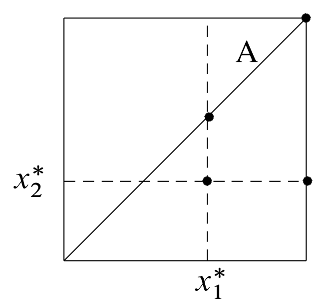

Figure 6.2. A card game

To see this, consider the diagram in figure 6.2. Because x_1 and x_2 are independently distributed, the pair (x_1, x_2) is uniformly distributed in the unit square. The case $P[x_1 > x_1^*, x_2 > x_1]$ is the little triangle labeled "A," which has area $(1 - x_1^*)^2/2$. We thus have

$$P[x_1 > x_1^*, x_2 > x_2^*, x_2 < x_1] = (1 - x_1^*)(1 - x_2^*) - \frac{(1 - x_1^*)^2}{2}.$$

To evaluate $P[x_1 > x_1^*, x_2 > x_2^*, x_2 > x_1]$ when $x_1^* < x_2^*$, refer to Figure 6.3. Calculating the area of trapezoid A representing the case $P[x_1 > x_1^*, x_2 > x_2^*, x_2 > x_1]$, we get

$$P[x_1 > x_1^*, x_2 > x_2^*, x_1 < x_2] = (1 - x_1^*)(1 - x_2^*) - \frac{(1 - x_2^*)^2}{2}.$$

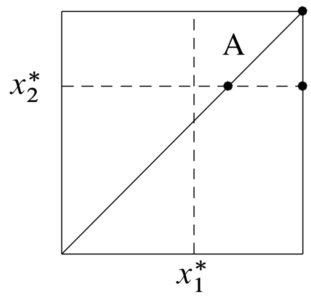

Figure 6.3. A card game II

Suppose $x_1^* > x_2^*$. The payoff to player 1 is then

$$\begin{aligned}\pi &= -x_1^* + (1 - x_1^*)x_2^* - 6\frac{(1 - x_1^*)^2}{2} \\ &\quad +6\left[(1 - x_1^*)(1 - x_2^*) - \frac{(1 - x_1^*)^2}{2}\right] \\ &= 5x_1^* - 5x_2^* - 6x_1^{*2} + 5x_1^* x_2^*.\end{aligned}$$

The first-order condition on x_2^* is then $-5 + 5x_1^* = 0$, so $x_1^* = 1$. The first-order condition on x_1^* is $5 - 12x_1^* + 5x_2^* = 0$, so $x_2^* = 7/5$, which is impossible.

Thus, we must have $x_1^* < x_2^*$. The payoff to player 1 is then

$$-x_1^* + (1 - x_1^*)x_2^* - 6\left[(1 - x_1^*)(1 - x_2^*) - \frac{(1 - x_2^*)^2}{2}\right] + 6\frac{(1 - x_2^*)^2}{2},$$

which reduces to

$$5x_1^* - 5x_2^* - 7x_1^* x_2^* + 6x_2^{*2}.$$

The first-order condition on x_1^* gives $x_2^* = 5/7$ and the first-order condition on x_2^* then gives $x_1^* = 25/49$. Note that we indeed have $x_1^* < x_2^*$. The payoff of the game to player 1 is then

$$5\frac{25}{49} - 5\frac{5}{7} + 6\left(\frac{5}{7}\right)^2 - 7\left(\frac{25}{49}\right)\left(\frac{5}{7}\right) = -\frac{25}{49}.$$

6.13 Cheater-Inspector

There are n rounds in a game between an inspector and a "taker." The taker can cheat in any round and the inspector can inspect in any round. If the taker cheats without getting inspected, the game stops and she gains 1 in that period and in every remaining period. If the taker is inspected while cheating, the game stops and she is fined a in that period and in every remaining period. If the taker is honest, she receives b in that round from the inspector if inspected and nothing if not inspected and the game goes on. The game is zero-sum (that is, whatever the taker gets the inspector loses). Let g_n be the payoff of game of length $n > 0$ and let $g_o = 0$. Then, for any n, we have the game in the diagram. Find the payoffs to the players in a Nash equilibrium.

	trust	inspect
cheat	n	$-na$
honest	g_{n-1}	$b + g_{n-1}$

6.14 The Vindication of the Hawk

Chicken (also known as the hawk-dove game) is a two-player game in which each player can either attack (A) or remain peaceful (P). Suppose at the start of the game, each player has one util of good stuff. If both players remain peaceful, they each get to consume their stuff. If one is peaceful and the other attacks, the attacker takes the other's stuff. But if both attack, they each lose $a > 0$ utils. This gives us the normal form matrix in the diagram. Show that there is a unique Nash equilibrium, players use completely mixed strategies and the payoff of the game to the players *increases* when the potential loss from conflict, a, increases.

	A	P
A	$-a,-a$	2,0
P	0,2	1,1

6.15 Characterizing 2×2 Normal Form Games I

We say a normal form game is *generic* if no two payoffs for the same player are equal. Suppose $A = (a_{ij})$ and $B = (b_{ij})$ are the payoff matrices for Alice and Bob, so the payoff to Alice's strategy s_i against Bob's strategy t_j is a_{ij} for Alice and b_{ij} for Bob. We say two generic 2×2 games with payoff matrices (A, B) and (C, D) are *equivalent* if, for all $i, j, k, l = 1, 2$:

$$a_{ij} > a_{kl} \text{ if and only if } c_{ij} > c_{kl}$$

and

$$b_{ij} > b_{kl} \text{ if and only if } d_{ij} > d_{kl}.$$

In particular, if a constant is added to the payoffs to all the pure strategies of one player when played against a given pure strategy of the other player, the resulting game is equivalent to the original.

Show that equivalent 2×2 generic games have the same number of pure Nash equilibria and the same number of strictly mixed Nash equilibria. Show also that every generic 2×2 game is equivalent to either the prisoner's dilemma (§3.11), the battle of the sexes (§3.9), or the hawk-dove (§3.10). Note that this list does not include throwing fingers (§3.8), which is not generic.

	L	R
U	a_1, a_2	b_1, b_2
D	c_1, c_2	d_1, d_2

To solve this problem, we refer to the figure in the diagram. first-order the strategies so the highest payoff for player 1 is a_1. Second, add constants so that $c_1 = d_1 = b_2 = d_2 = 0$. Because the game is generic, $a_1 > 0$ and either $a_2 > 0$ (case I) or $a_2 < 0$ (case II). Third, explain why only the signs of c_2 and b_1, rather than their magnitudes, remain to be analyzed. If either is positive in case I, the game has a unique equilibrium found by the iterated elimination of dominated strategies and is equivalent to the prisoner's dilemma. The same is true in case II if either $b_1 > 0$ or $c_2 < 0$. The only remaining case I situation is $b_1, c_2 < 0$, which is equivalent to the battle of the sexes, with two pure- and one mixed-strategy equilibria. The only remaining case II is $b_1 < 0, c_2 > 0$, which is equivalent to hawk-dove and there is a unique mixed-strategy equilibrium.

6.16 Big John and Little John Revisited

Find the mixed-strategy Nash equilibrium to the simultaneous-move Big John and Little John game discussed at the end of section 3.1.

6.17 Dominance Revisited

Show that if a game has a solution by the iterated elimination of strongly dominated strategies (§4.1), then this solution is the only Nash equilibrium of the game. Hint: Use the fundamental theorem to show that each strongly dominated strategy has weight 0 in a mixed-strategy Nash equilibrium.

6.18 Competition on Main Street Revisited

In Competition on Main Street (§5.2), you showed that there is no pure-strategy equilibrium with three agents. Suppose that general stores can only be set up at locations $0, 1/n, \ldots, (n-1)/n, 1$ (multiple stores can occupy the same location).

a. Let $\pi(x, y, z)$ be the payoff to the agent choosing location x when the other two agents choose y and z. Find an expression for $\pi(x, y, z)$.
b. Show that for $n = 4$ there is a mixed-strategy Nash equilibrium in which each agent locates at points $1/4$ and $3/4$ with probability $1/7$ and point $1/2$ with probability $5/7$.
c. Show that for $n = 6$ there is a mixed-strategy Nash equilibrium in which each agent locates at points $1/3$, $1/2$, and $2/3$ each with probability $1/3$.
d.* Show that for $n = 10$ there is no mixed-strategy equilibrium in which all agents locate within one location of the center, but there is one in which they locate within two locations of the center. Show that locating at 3/10, 2/5, 1/2, 4/5, and 7/10 with equal probability is such an equilibrium.
e.* If you have the appropriate mathematical software (e.g., Mathematica or Maple), or if you have a long weekend with nothing to do, find mixed-strategy equilibria for $n = 12, 14, 16$. Hint: In each case there are five locations that are occupied with nonzero probability and the probabilities are symmetric around $n/2$.

6.19 Twin Sisters Revisited

In section 5.15, a mother tells each of her twin daughters to ask for a certain whole number of dollars, at least 1 and at most 100. If the total of the two amounts does not exceed 101, each will have her request granted. Otherwise each gets nothing. What will the sisters do?

You probably answered that both sisters would ask for \$50, even though this is not a Nash equilibrium. However, if one sister is pretty sure the other will bid \$50, she herself might be tempted to bid \$51. With mixed strategies available, we can find a Nash equilibrium that captures our intuition that bidding \$50 will almost always be the observed behavior.

Let us write s_x for the pure strategy "bid x." Suppose both sisters use the mixed strategy $\sigma = ps_x + (1-p)s_y$, where $p \in (0, 1)$ and (σ, σ) is a Nash equilibrium. In any Nash equilibrium we must have $x + y = 101$, so we can assume that $x < y = 101 - x$. Check that because the payoffs to s_x and s_y must be equal, we have $x = py$. Show that any σ satisfying the previous two conditions is indeed a Nash equilibrium; that is, show that no pure strategy s_z has higher payoff against σ that σ has against σ (Hint: consider separately the cases $z < x$, $x < z < 51$, $51 < z < y$, and $y < z$).

It is easy to see that the payoff to the Nash equilibrium with $x < y = 101 - x$ and $x = py$ is simply x per sister. Thus, though there are many Nash equilibria, the highest payoff is the one in which $x = 50$, $y = 51$, and $p = 50/51$, which is practically unity. So both sisters will ask for \$50 most of the time, as our intuition suggested to us.

But why should the highest-payoff Nash equilibrium actually be the one that the sisters choose? Could they not get "locked into" an inferior Nash equilibrium, say where $x = 10$? The answer is: to this point, we have no way of answering this question. But suppose the way that sisters play this game is a sort of "social convention" that people learn and suppose further that there is some sort of social process whereby superior conventions grow in frequency and inferior ones contract. Then, perhaps, the $x = \$50$ solution might come to be established in society.

6.20 Twin Sisters: An Agent-Based Model

We can use the procedures described in section 4.20 to create an agent-based model of Twin Sisters (§5.15). I wrote a program (in the Pascal programming language) with 200 agents, each of whom is given a random strategy s_i, where i is an integer between 0 and 101, so there were approximately 10 agents with each possible strategy. The were randomly paired for 40,000 generations, each consisting of a single period of play. In each generation, 10 players "died" and were replaced by the "offspring" of 10 other players, the probabilities of dying and of reproducing being proportional to the player's current score in the game. Moreover, 10% of the new agents were given a new, random bid ("mutation"). The results are shown in figure 6.4. Note that at the beginning of the simulation, s_{49}, s_{50}, and s_{51} represent only a few % of the population, but after 10,000 generations, they represent almost the whole population. By the end of the simulation, s_{49} has dwindled, leaving only the two strategies we intuitively expected to be played, s_{50} and s_{51} (plus stray mutations, not shown in the figure).

Note also that the relative proportions of the two remaining strategies are approximately 50 to 1, as expected in a Nash equilibrium.

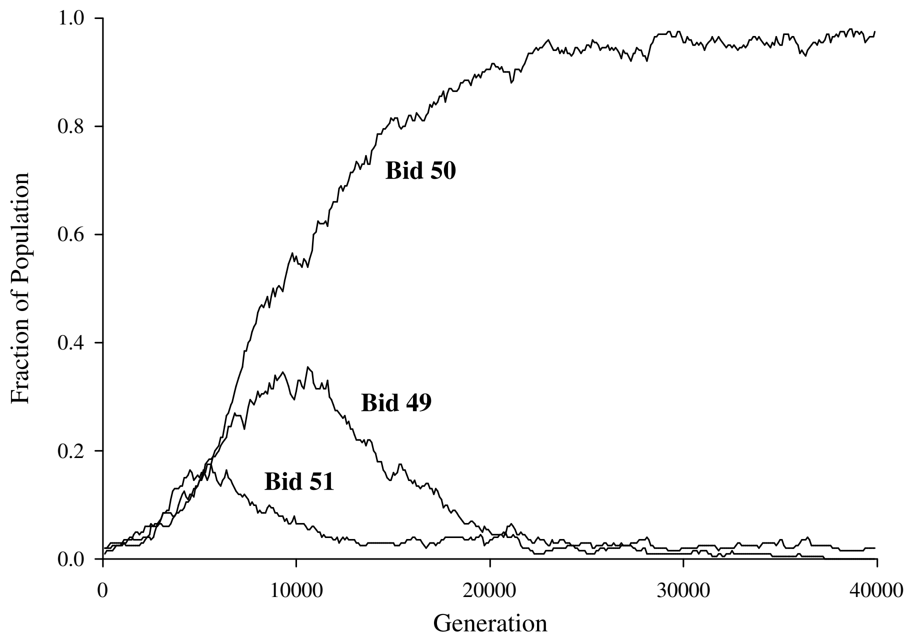

Figure 6.4. Twin sisters: an agent-based model

In chapters 10 and 12 we deal in depth with dynamical and evolutionary models, but some simple calculations suggest that the more unequal the two bids, say s_x and s_{101-x}, the smaller the minimum fraction q of "invaders" using s_{50} need be to displace a population composed of a fraction $1-q$ of agents playing the mixed strategy using s_x and s_{101-x}. As an exercise, you can show that $q = x(51-x)/50(101-2x)$, which increases almost linearly from $x = 0$ to $x = 45$ but then increases sharply to unity as x moves from 45 to 50.

This example supports the evolutionary notion that the Nash concept is justified not by the fact that a few wise, rational agents will play Nash equilibria, but because many intellectually challenged agents (the agents in our model could do absolutely nothing but (a) play their genetically inherited strategies and (b) reproduce) could dynamically settle on one of the many Nash equilibria of the game. As we shall see in chapter 10, the equilibrium points of evolutionary dynamics are always Nash equilibria, although not all Nash equilibria are equilibrium points of an evolutionary dynamic and not all equilibrium points of an evolutionary dynamic are stable in the appropriate sense.

6.21 One-Card, Two-Round Poker with Bluffing

Alice and Bob start by each putting \$2 into the "pot." Alice is dealt a card, which with equal probability is either H (high) or L (low). After looking at her card, which Bob cannot see, she either raises or folds. If she folds, the game is over and Bob takes the pot. If she raises, she must put an additional \$2 into the pot and Bob must now either stay or fold. If Bob folds, the game is over and he loses the pot. If he stays, he must put an additional \$2 into the pot to meet Alice's previous bet and Alice has another turn. Alice must again raise or fold. If she folds, she loses the pot and if she plays, she must put another \$2 into the pot and Bob has a final turn, in which he must either fold or stay. If Bob folds, he loses the pot and the game is over. If he stays, he must put an additional \$2 into the pot and Alice must show her card. If it is H, she wins the pot and if it is L, she loses the pot.

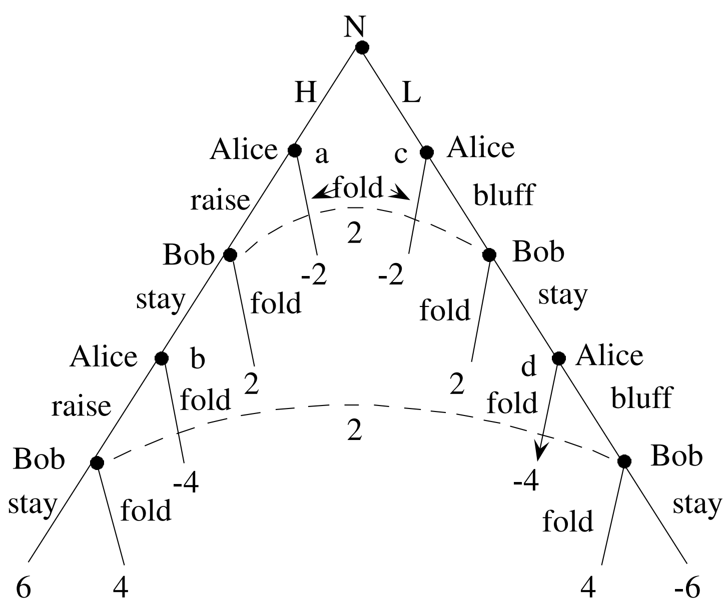

Figure 6.5. One-card two-round poker with bluffing

It is easy to see that Bob has three pure strategies: ss (stay,stay), sf (stay,fold), and f (fold). Alice has nine strategies: $rrbb$ (raise,raise on H, and bluff,bluff on L), $rrbf$ (raise,raise on H, and bluff,fold on L), rrf (raise,raise on H, and fold on L), $rfbb$ (raise,fold on H, and bluff,bluff on L), $rfbf$ (raise,fold on H, bluff,fold on L), fbb (fold on H, bluff,bluff on L), fbf (fold on H, bluff,fold on L), rff (raise,fold on H, fold on L), and ff (fold on H, fold on L).

Show that Alice has only three weakly undominated pure strategies and find her best-response mixed strategy. Then find the best-response mixed strategy for Bob. This gives the Nash equilibrium for the game.

6.22 An Agent-Based Model of Poker with Bluffing

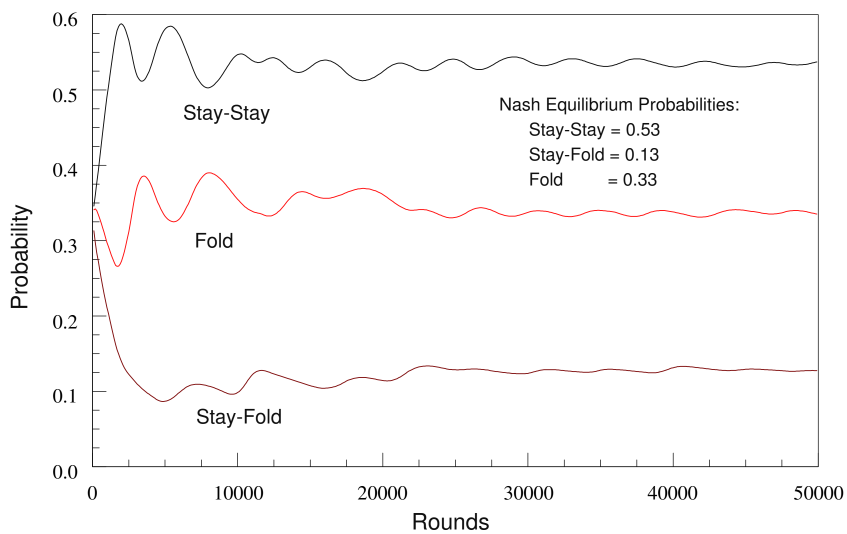

Figure 6.6. An Agent-based model of one-card two-round poker with bluffing

We can create an agent-based model of one-card two-round poker with bluffing by creating in the computer silicon creatures with very little information processing capacity (none, in fact). The creature's genome consists of a mixed strategy (that is, a probability distribution over the three nondominated strategies) for Alice and similarly for Bob. In this model, I created 200 players of each type and assigned them pure strategies randomly. In each period of play, partners are randomly assigned and every 100 periods we allow reproduction to take place. In this model, reproduction consisted in killing off the player with the lowest score and allowing the player with the highest score to reproduce, with mutation in the genome at rate 2%. The simulation ran for 50,000 periods. The results of one run of the simulations for the distribution of Bob types in the economy are shown in figure 6.6. Note that after 25,000 periods, the frequency of each strategy has settled down to the theoretically predicted equilibrium value.

6.23 Trust in Networks

Consider a network of many traders who are randomly paired to play a one-shot (that is, played only once) prisoner's dilemma in which each receives -1 if they both defect, each receives 1 if they both cooperate and a defector receives 2 when playing against a cooperator, who receives -2. There are three types of agents: defectors defect unconditionally against all partners; trusters cooperate unconditionally with all partners; and inspectors monitor an imperfect signal indicating whether or not one's current partner defects against cooperators. The signal correctly identifies a defector with probability $p > 1/2$ and correctly identifies a non-defector with the same probability p. The inspector then refuses to trade with a partner who is signalled as a defector and otherwise plays the cooperate strategy. An agent who does not trade has payoff 0. The payoff matrix for a pair of agents has the normal form shown in figure 6.7.

	Inspect	Trust	Defect
Inspect	p^2 p^2	p p	$-2(1-p)$ $2(1-p)$
Trust	p p	1 1	-2 2
Defect	$2(1-p)$ $-2(1-p)$	2 -2	-1 -1

Figure 6.7. The inspect-trust-defect game

Think of a "strategy" in this network as a fraction α of inspectors, a fraction β of trusters and a fraction $1-\alpha-\beta$ of defectors. A Nash equilibrium is a population composition (α, β) that is a best response to itself.

It is easy to show that there are no pure-strategy Nash equilibria and for $p \geq \sqrt{3}-1 \approx 3/4$, there are no Nash equilibria involving only two types of players. Use section 3.6 to prove that there exists a unique completely mixed Nash equilibrium for $p > 5/7$ and show that it is unique.

6.24 El Farol

In Santa Fe there is nothing to do at night but look at the stars or go to the local bar, El Farol. Let us define the utility of looking at the stars as 0 and let the cost of walking over to the bar be 1. Suppose the utility from being at the bar is 2 if there are fewer than three people at the bar and 1/2 if there are three or more people at the bar. For a given Nash equilibrium, we define the *social surplus* provided by the bar to be the sum of the payoffs to all the residents.

Suppose there are three people in Santa Fe, so there are three pure-strategy Nash equilibria, in each of which two people go to the bar and one watches the stars. The average payoff per player in each of these is 2/3. There is also a unique symmetric mixed-strategy Nash equilibrium (that is, each player uses the same mixed strategy) in which each resident goes to the bar with probability $\sqrt{(2/3)} \approx 81.65\%$. To see this, let p be the probability of going to the bar for each resident. The payoff to not going to the bar and the payoff to going to the bar must be equal. To find p, note that the probability that the other two people go to the bar is p^2, so the expected payoff to going to the bar is

$$2(1 - p^2) + \frac{1}{2}p^2 - 1 = 0,$$

the solution to which is $p = \sqrt{(2/3)}$.

Note that in this equilibrium the average payoff is 0: the bar might as well not exist!

To generalize the problem, suppose for each player $i = 1, 2, 3$ the cost of walking to the bar is c_i, the payoff when there are fewer than three people in the bar is a_i and the payoff otherwise is b_i, where $0 < b_i < c_i < a_i$. Now, if there is any mixed-strategy equilibrium, it is unique and once again bar might as well not exist. To see this, let p_i be the probability of player i going to the bar, for $i = 1, \ldots, 3$. In a mixed strategy-equilibrium, the payoff for each player to going to the bar and staying home must be the same. It is easy to show that this is equivalent to

$$p_i p_j = \frac{a_k - c_k}{a_k - b_k} \qquad i \neq j \neq k \neq i.$$

Let α_k be the right-hand side of this equation. We can solve the resulting three equations, getting $p_i = \sqrt{\alpha_j \alpha_k / \alpha_i}$. The conditions for a mixed-strategy equilibrium are thus $\alpha_i \alpha_j < \alpha_k$ for $i \neq j \neq k \neq i$. We conclude that, if the costs and benefits of the bar are not too dissimilar for the three players, the mixed-strategy equilibrium exists. Otherwise, one resident must always stay home. The only equilibrium in which there is a positive payoff from the bar's existence is if at least one resident stays home.

We can generalize the problem to n people, in which case the bar still might as well not exist, provided the equilibrium is completely mixed. However, you can show that if El Farol charges an appropriate entry fee, the payoff to both the bar and its clients can be strictly positive.

6.25 Decorated Lizards

The side-blotched lizard *Uta stansburiana* has three distinct male types: orange-throats, blue-throats, and yellow-striped. The orange-throats are violently aggressive, keep large harems of females (up to seven) and defend large territories. The blue-throats are less aggressive, keep small harems (usually three females) and defend small territories. The yellow-stripes are very docile but they look like females, so they can infiltrate another male's territory and secretly copulate with the females. Field researchers note that there is regular succession from generation to generation, in which orange-throated males are a majority in one period, followed by a majority of yellow-striped males, who are followed in turn by a majority of blue-throated males and finally by a new majority of orange-throated males, thus completing the cycle.

This cycle occurs because the orange-throats have so large a territory and so large a harem that they cannot guard effectively against the sneaky yellow-striped males, who mix in with the females and because they look a lot like females, go undetected by the orange-throats, who could easily detect the bright blue-throat males. The yellow-striped males thus manage to secure a majority of the copulations and hence sire lots of yellow-striped males, who are very common in the next period. When yellow-stripes are very common, however, the males of the blue-throated variety benefit, because they can detect and eject the yellow-stripes, as the blue-throats have smaller territories and fewer females to monitor. The blue-throat males thus have the greatest number of male offspring in the next period, which is thus dominated by blue-throat males. When the blue-throats predominate, the vigorous orange-throats eject them from their territories and hence they come to dominate the succeeding period, because they acquire the blue-throat harems and territories. Thus there is a recurring three-period cycle in which each type of male dominates in one period, only to be outdone by a different male type in the succeeding period.

The game underlying this is the familiar children's game rock, paper, and scissors, with the payoff structure as in the diagram. Note that just as in the lizard case, each "type" (rock, paper, scissors), receives 0 payoff playing against itself, but is superior to one of its two dissimilar adversaries and is inferior to the other of its dissimilar adversaries (yellow-striped beats orange-throat but is beaten by blue-throat; orange-throat beats blue-throat but is beaten by yellow-striped; blue-throat beats yellow-striped but is beaten by orange-throat). Show that the only Nash equilibrium to this game is the mixed-strategy equilibrium in which each strategy is played with equal probability.

	Orange Throat	Yellow Striped	Blue Throat
Orange Throat	0,0	−1,1	1,−1
Yellow Striped	1,−1	0,0	−1,1
Blue Throat	−1,1	1,−1	0,0

After you have learned how to model game dynamics, we will return to this problem and show that under a replicator dynamic, the male lizard population does indeed cycle among the three forms in successive breeding periods (§12.14).

6.26 Sex Ratios as Nash Equilibria

Most organisms that employ sexual reproduction have two sexes: male and female. The fraction of a female's offspring that are female is determined by genetic factors and hence is heritable. In many species (e.g., most animals), the fraction is almost exactly 1/2, even if the viabilities of males (σ_m) and females (σ_f), the probability that they mature to the point of sexual reproduction, are very different. Why is this the case?

To streamline the process of solving this problem, suppose all females breed simultaneously and their offspring constitute the next generation of birds (that is, birds live for only one breeding period). Unless otherwise stated, you should assume (a) females "choose" a ratio u of sons to daughters born that maximizes the expected number of their genes among their grandchildren; (b) each female produces c offspring; (c) males and females contribute an equal number of genes to their offspring; (d) all males are equally likely to sire an offspring; (e) there is random mating in the next generation; and (f) the next generation is so large that no single female can affect the ratio v of males to females in the next generation.

First we show that $u = 1/2$ in equilibrium; that is, a female produces half sons and half daughters. Call the birds surviving to maturity in the next generation the "breeding pool." Let s and d be the number of sons and daughters in the breeding pool. Then $\alpha = d/s$ is the expected fraction of a female's offspring sired by any given male in the breeding pool. We then have

$$\alpha = \frac{d}{s} = \frac{\sigma_f(1-v)}{\sigma_m v}. \tag{6.3}$$

We now write an expression for $f(u, v)$, the number of grandchildren of a female, in terms of α and the other parameters of the problem (u, v, σ_f, σ_m, and c). We have

$$f(u, v) = \sigma_f(1-u)c^2 + \sigma_m u c^2 \alpha. \tag{6.4}$$

To understand this expression, note that $\sigma_f(1-u)c$ is the number of daughters who survive to maturity and so $\sigma_f(1-u)c^2$ is the number of grandchildren born to daughters. Similarly, $\sigma_m uc$ is the number of sons and $\sigma_m uc(c\alpha)$ is the number of grandchildren sired by sons.

Substituting equation (6.3) into equation (6.4) and simplifying, we get

$$f(u, v) = c^2\sigma_f \left\{1 + u\left(\frac{1-2v}{v}\right)\right\}.$$

If we now choose u to maximize $f(u, v)$, we see that the only Nash equilibrium occurs when $u = v$. Thus, if $v \neq 1/2$, there cannot be a mixed-strategy equilibrium: if the fraction of males in the population is less than 50%, each female should produce all males (that is, set $u = 1$) and if the fraction of males in the population is greater than 50%, each female should produce all females (that is, set $u = 0$). The only possible Nash strategy is therefore $u = v = 1/2$, because such a strategy must be symmetric (the same for all agents) and mixed (because all pure strategies are clearly not Nash).

Suppose now that there are n females and n is sufficiently small that a single female's choice *does* affect the ratio of daughters to sons. We can still show that an equal number of daughters and sons remains a Nash equilibrium. It is easy to check that (6.3) becomes

$$\alpha = \frac{d}{s} = \frac{\sigma_f[n-u-(n-1)v]}{\sigma_m[(n-1)v+u]}.$$

The number of grandchildren as expressed in (6.4) then becomes

$$\begin{aligned} f(u,v) &= \sigma_f(1-u)c^2 + \sigma_m u c^2 \frac{\sigma_f[n-u-(n-1)v]}{\sigma_m[(n-1)v+u]} \\ &= \frac{c^2\sigma_f}{(n-1)v+u}\{-2u^2 - u[2(n-1)v-(n+1)] + (n-1)v\}. \end{aligned}$$

The first-order condition on u for maximizing $f(u,v)$ then gives

$$2(n-1)v = n+1-4u.$$

In a symmetric equilibrium, we must have $u = v$, which implies $u = v = 1/2$.

Now suppose that instead of only breeding once, a fraction δ_m of breeding males and δ_f of breeding females die in each period and the rest remain in the mating pool. The expression for the equilibrium ratio of males to females is derived as follows. Let m be the number of males and let n be the number of females in the first period. Then the ratio α of females to males in the breeding pool in the next period is given by

$$\alpha = \frac{d+n(1-\delta_f)}{s+m(1-\delta_m)} = \frac{\sigma_f cn(1-v)+n(1-\delta_f)}{\sigma_m cnv + m(1-\delta_m)}. \tag{6.5}$$

The number of grandchildren of one female who has fraction u of males and $1-u$ of females, when the corresponding fraction for other breeding females is v, is given by

$$f(u,v) = c^2\left[\sigma_f(1-u)+\sigma_m u\alpha\right] = c^2\left\{1+u\left[\frac{\sigma_m}{\sigma_f}\alpha - 1\right]\right\}.$$

Hence, a mixed-strategy Nash equilibrium requires

$$\alpha = \frac{\sigma_f}{\sigma_m}. \tag{6.6}$$

Solving (6.5) and (6.6) and simplifying, we get

$$v = \frac{1}{2}\left[1 - \frac{\sigma_f \gamma(1-\delta_m) - \sigma_m(1-\delta_f)}{\sigma_m \sigma_f c}\right], \tag{6.7}$$

where we have written $\gamma = m/n$. But in the second period, m is simply the denominator of (6.5) and n is the numerator of (6.5), so (6.6) implies $\gamma = m/n = \sigma_m/\sigma_f$. Substituting this expression for γ in (6.7), we get

$$v = \frac{1}{2}\left[1 - \frac{\delta_f - \delta_m}{\sigma_f c}\right],$$

from which the result follows. Note that this ratio remains 1/2 if $\delta_f = \delta_m$.

Finally, suppose the species is haplodiploid (many bee species are). This means that males have only one copy of each gene, which they get from their mother (that is, males come from unfertilized eggs), whereas females have two copies of each gene, one from each parent. We will find the equilibrium ratio of daughters to sons assuming birds live for one breeding period and females maximize the number of copies of their genes in their grandchildren. For a female who has fraction u of sons and $1-u$ of daughters, when the corresponding fraction for other breeding females is v, the fraction of genes in daughters is $c(1-u)/2$ and the fraction in sons is cu. The number of genes (normalizing the mother's gene complement to unity) in daughters of daughters is $c^2(1-u)(1-v)/4$, the number of genes in sons of daughters is $c^2(1-u)v/2$ and the number of genes in daughters of sons is $c^2 u\alpha(1-v)$. None of the female's genes are in sons of sons, because only the mother passes genetic material to her sons. The number of genes in the mother's grandchildren is the sum of these three components, which simplifies to

$$f(u,v) = c^2\left\{\frac{1+v}{4} - u\left[\frac{1+v}{4} - (1-v)\alpha\right]\right\},$$

so we must have

$$\alpha = \frac{1+v}{4(1-v)}. \tag{6.8}$$

But by our assumption that individuals live for only one breeding period, (6.3) still holds. Solving (6.3) and (6.8) simultaneously and defining $\nu = \sigma_f/\sigma_m$, we get

$$v = \frac{1 + 8\nu \pm \sqrt{32\nu + 1}}{2(4\nu - 1)},$$

where the sign of the square root is chosen to ensure $0 < v < 1$. This implies that, for instance, if $\sigma_f = \sigma_m$, then $v \approx 0.54$; that is, the ratio of daughters to sons should be only slightly biased toward males.

6.27 A Mating Game

Consider a mating system in which there are males and females, 50% of each sex being *hierarchical* (H) and the other half *egalitarian* (E). When a male meets a female to mate, their sex is visible, but neither knows the other's H/E type. There are two mating strategies: *forward* (F) and *reserved* (R). Females prefer their partners to be reserved, but males prefer to be forward. In addition, when a pair of hierarchicals meet, they both prefer that one be forward and the other reserved, but when a pair of egalitarians meet, they both prefer to play the same strategy, both forward or both reserved. The payoffs are depicted in figure 6.8.

	F	R
F	0,0	2,1
R	1,2	0,0

H Meets H

	F	R
F	0,2	2,0
R	1,0	0,1

H Meets E

	F	R
F	1,0	0,1
R	0,2	2,0

E Meets H

	F	R
F	1,2	0,0
R	0,0	2,1

E Meets E

Figure 6.8. Mating game payoffs, where the female is the row player

There are four pure strategies: FF (forward if H, forward if E), FR (forward if H, reserved if E), RF (reserved if H, forward if E), RR (reserved if H, reserved if E). A mixed strategy for a female is a pair of probabilities (α_H, α_E), where α_H is the probability of being forward when she is H and α_E is the probability of being forward when she is E. A mixed strategy for a male is a pair of probabilities (β_H, β_E), where β_H is the probability of being forward when he is H and β_E is the probability of being forward when he is E. Find all Nash equilibria of the game.

6.28 Coordination Failure

Find the unique mixed Nash equilibrium of the game in the diagram. Show that if either player adopts any strategy other than his unique Nash strategy, the optimal response by the other player will result in a superior outcome for both. In this case, then, the Nash equilibrium is the worst of all possible worlds.

	L	M	R
U	0,0	50,40	40,50
C	40,50	0,0	50,40
D	50,40	40,50	0,0

6.29 Colonel Blotto Game

Colonel Blotto and his adversary, the Folks' Militia each try to occupy two posts by properly distributing their forces. Colonel Blotto has four regiments and the Militia has three regiments. If Colonel Blotto has more regiments than the enemy at a post, Colonel Blotto receives the enemy's regiments plus one (that is, one is the value of occupying the post). If Colonel Blotto has fewer regiments at a post than the enemy, he loses one plus the number of regiments he has at the post. A draw gives both sides 0. The total payoff is the sum of the payoffs at the two posts. Show that Colonel Blotto has five pure strategies and the Folks' Militia has four pure strategies. Write the payoff matrix and find the Nash equilibria of the game.

6.30 Number Guessing Game

Bob picks a number from 1 to 3. Alice tries to guess the number. Bob responds (truthfully!) by saying "high," "low," or "correct." The game continues until Alice guess correctly. Bob takes a number of dollars from Alice equal to the number of guesses Alice took.

The game is determined in the first two rounds. Let us write Alice's strategies as (g h l), for "first guess g, if high guess h and if low guess l." If a high guess is impossible, we write (1 x l) and if a low guess is impossible, we write (3 h x). For instance, (1x3) means "first choose 1 and if this is low, then choose 3." Write the payoff matrix and find the Nash equilibria of the game.

6.31 Target Selection

There are n targets whose military values are $a_1, \ldots, a_n$, where $a_1 > a_2 > \ldots > a_n > 0$. Attacker has one attacking unit to allocate to one of the n targets and defender has one unit to allocate to the defense of the targets. If target k is attacked and is undefended, it will be captured, with the value a_k going to attacker. If target k is defended, it has a probability p of being successfully held by defender, so the expected payoff to attacker is $(1-p)a_k$.

Show that there is some m such that targets $1, \ldots, m$ are attacked and defended with positive probability and targets $m+1, \ldots, n$ are neither attacked nor defended.

6.32 A Reconnaissance Game

Attacker can either attack with all its forces (strategy 1) or attack with part of its forces, leaving the remainder as reserves and rear guards in case its forces are outflanked (strategy 2). Defender has the same two strategy options. The payoff to attacker if attacker uses strategy i and defender uses strategy j is a_{ij}. We assume it is best for attacker to use the same strategy as defender; that is, $a_{11} > a_{21}$ and $a_{22} > a_{12}$.

Attacker can also send out a reconnaissance force, at cost c, to find out defender's strategy. This will surely work, unless defender takes countermeasures at cost d (these countermeasures must be taken without knowing whether Attacker will actually reconnoiter), in which case reconnaissance will fail. Suppose the game is zero-sum, with attacker payoff given by

$$A = [a_{ij}] = \begin{bmatrix} 48 & 24 \\ 12 & 36 \end{bmatrix} \qquad \text{with } c = 9 \text{ and } d = 7.$$

Find the Nash equilibria.

6.33 Attack on Hidden Object

Alice has a bomb that she wants to drop on Bob's country. She can carry the bomb in one of two identical bombers P (protected) and F (flank). Bob can prevent the damage by destroying the bomber containing the bomb.

The two bombers fly in formation, so to attack P, Bob must fly past F, in which case he runs the risk α, with $0<\alpha<1$, of being shot down before engaging P. Once Bob has engaged his target (whether F or P), he can destroy it with probability β, with $0<\beta 1$. Thus, in any attack on P, if F is intact, the probability of destroying P is $\gamma = (1-\alpha)\beta$.

Suppose Bob has enough fuel to attack the bombers twice and hence, two chances to hit a target and destroy the valued object. Alice has two strategies: load the bomb in F and load the bomb in P. Bob has four strategies: attack F both times, attack P both times, attack F the first time and P the second time and vice versa. The understanding is that if the first attack was successful, the second attack is directed against the remaining target, whatever the strategy used. Write normal form and find the Nash equilibria.

6.34 Two-Person, Zero-Sum Games

A *zero-sum game* is, appropriately enough, a game in which the sum of the payoffs to all the players is 0. Von Neumann and Morgenstern (1944), who launched modern game theory, lay great stress on zero-sum games and, indeed, defined equilibrium in a way that works only for two-person zero-sum games. Nash had not yet invented the equilibrium concept that bears his name; that came in 1950.

Suppose the payoff to player 1 is $\pi(\sigma,\tau)$ when player 1 uses σ and player 2 uses τ, so the payoff to player 2 is $-\pi(\sigma,\tau)$. Von Neumann and Morgenstern defined (σ^*,τ^*) to be an equilibrium of the two-person, zero-sum game if σ^* maximizes $\min_\tau \pi(\sigma,\tau)$ and τ^* minimizes $\max_\sigma \pi(\sigma,\tau)$. They showed that this *maximin solution* satisfies

$$\pi(\sigma^*,\tau^*) = \max_\sigma \min_\tau \pi(\sigma,\tau) = \min_\tau \max_\sigma \pi(\sigma,\tau). \tag{6.9}$$

It is easy to show that *a strategy profile in a two-person, zero-sum game is a Nash equilibrium if and only if it is a maximin solution*. This implies, in particular, that *all Nash equilibria of a two-person, zero-sum game have the same payoffs*.

To prove a Nash equilibrium is a maximin solution, suppose (σ^*,τ^*) is a Nash equilibrium. Then, for all σ,τ, we have

$$\begin{aligned}\max_\sigma \pi(\sigma,\tau^*) &\geq \min_\tau \max_\sigma \pi(\sigma,\tau) \geq \pi(\sigma^*,\tau^*)\\ &\geq \max_\sigma \min_\tau \pi(\sigma,\tau) \geq \min_\tau \pi(\sigma^*,\tau).\end{aligned}$$

The first inequality is obvious, the second follows from

$$\min_{\tau} \max_{\sigma} \pi(\sigma, \tau) \geq \min_{\tau} \pi(\sigma^*, \tau) = \pi(\sigma^*, \tau^*),$$

the third follows from

$$\max_{\sigma} \min_{\tau} \pi(\sigma, \tau) \leq \max_{\sigma} \pi(\sigma, \tau^*) = \pi(\sigma^*, \tau^*),$$

and the fourth inequality is obvious. But the first and third terms must then be equal, because (σ^*, τ^*) is Nash and similarly for the third and fifth terms. Thus, they are all equal, so (σ^*, τ^*) is maximin.

To show that a maximin solution is a Nash equilibrium, suppose (σ^*, τ^*) is maximin. We know that the second equation in (6.9) holds because there exists a Nash equilibrium and we have already shown that a Nash equilibrium is maximin satisfying (6.9). But then we have

$$\pi(\sigma^*, \tau^*) \leq \max_{\sigma} \pi(\sigma, \tau^*) = \min_{\tau} \pi(\sigma^*, \tau) \leq \pi(\sigma^*, \tau^*).$$

This proves all three terms are equal, so (σ^*, τ^*) is Nash.

6.35 Mutual Monitoring in a Partnership

Two agents share a resource. One agent, whom we call the "taker," gets to use the resource and can either steal (S) or be honest (H) in the amount of resource used. The other agent, the "watcher," can monitor (M) or trust (T). We normalize the payoffs to the two players following the "cooperative" strategy (T, H) to be 0. Let b be the benefit to the taker from stealing and not getting caught, let p be the loss to the taker from getting caught stealing and let α be the probability of getting caught if the watcher monitors. Also, let c be the cost to the watcher of monitoring and let λ be the loss to the watcher if the taker steals and is not caught. We assume $b, p, \alpha, \lambda > 0$. We can normalize $b + p = 1$ (e.g., by dividing all of the payoffs to player 1 by $b + p$). The game matrix is then given in the diagram.

	T	M
H	0,0	0,$-c$
S	b,$-\lambda$	$b - \alpha$,$-\lambda(1 - \alpha) - c$

Let μ be the probability of monitoring in the watcher's mixed strategy and let σ be the probability of stealing in the taker's mixed strategy.

a. Prove that if $c < \alpha\lambda$ and $b < \alpha$, then there is a completely mixed-strategy Nash equilibrium with $\mu = b/\alpha$ and $\sigma = c/\alpha\lambda$. Show that the payoff to the taker is 0 and the payoff to the watcher is $-c/\alpha$.
b. Explain why the loss to the watcher depends only on c and α and not, for instance, on λ. Explain why the return to the taker does not depend on any of the parameters of the problem, so long as $c < \alpha\lambda$ and $b < \alpha$.
c. What are the Nash equilibria if one or both of the inequalities $c < \alpha\lambda$ and $b < \alpha$ are violated?

6.36 Mutual Monitoring in Teams

This is a continuation of the previous problem. Now suppose there are $n + 1$ agents, where agent $n + 1$ is the "taker," agents $1, \ldots, n$ being identical "watchers." Suppose each watcher has the same probability $\alpha > 0$ of detecting stealing if monitoring, the same cost $c > 0$ of monitoring and the same loss $\lambda > 0$ from an undetected theft. We care only about *symmetric equilibria*, in which all watchers choose the *same* probability μ of monitoring the taker.

Let $b < 1$ be the gain to the taker from stealing and define

$$\rho = \alpha\lambda(1 - b)^{\frac{n-1}{n}}.$$

Answer the following questions, assuming $b < \alpha\lambda$ and $c < \rho$.

a. Show that there is a mixed-strategy Nash equilibrium with the probability σ of stealing and the probability μ of monitoring given by

$$\sigma = \frac{c}{\rho} \qquad \mu = \frac{1 - (1 - b)^{\frac{1}{n}}}{\alpha}.$$

b. Show that the payoff to a watcher is now $-c/\alpha(1 - b)^{\frac{n-1}{n}}$. Why does this not depend on λ?
c. How does this solution change as the group size n increases? Why does the tragedy of the commons (that is, the free rider) result not hold in this case?

d. What would happen as n increases if, for some fixed λ^*, we wrote $\lambda = \lambda^*/n$? This formulation would be reasonable if a dishonest taker imposed a fixed cost on the group no matter what its size, the cost being shared equally by the watchers.

6.37 Altruism(?) in Bird Flocks

This is an application of the results of the previous problem. Consider a flock of n birds eating in a group. A cat can catch a bird if it can sneak up behind a nearby rock without being seen. Each bird has an incentive to let the other birds look out for the cat whereas it conserves all its resources for eating (studies show that birds dissipate a considerable amount of energy and lose a considerable amount of time looking out for enemies). Why then do birds actually look out for predators when they eat in flocks? Are they "altruistic"? Perhaps not.[1]

Suppose it takes the cat one second out in the open to reach the rock. If seen during that one second by even one bird, the birds will all fly off and the cat will lose $p \in (0,1)$ in wasted time. If the cat reaches the rock, it catches one of the birds for a gain of $b = 1 - p$.

If each bird looks up from eating every $k \geq 1$ seconds, it will see the cat with probability $1/k$. Thus, we can take $\alpha = 1$ and $\mu = 1/k$ in the previous problem (T corresponds to $k = \infty$). The cost to the bird of being caught is $\lambda = 1$ and the cost of looking up once is c. Prove the following, where we define $\mu = 1 - c^{1/(n-1)}$ and $\eta = 1 - c^{n/(n-1)}$.

THEOREM: There are three types of symmetric equilibria.

a. If $c > 1$, then no bird looks up and the cat stalks the birds with probability 1.
b. If $c < 1$ and $b > \eta$, there is a symmetric Nash equilibrium in which the cat stalks with certainty, and birds look up every $1/\mu$ seconds.
c. If $b < \eta$, then there is a symmetric Nash equilibrium where the cat stalks with probability $\sigma = c(1-b)^{-(n-1)/n}$ and the birds look up with probability $1/\mu$.

[1] A similar problem arises in modeling the foraging behavior of flocks of birds in patchy environments, because if one bird finds a patch of food, all get to eat their fill (Motro 1991; Benkman 1988).

6.38 The Groucho Marx Game

Alice and Bob ante an amount $a \geq 0$ and cards numbered from 1 to n are placed in a hat. The players draw one card each, each observing his own but not the other's. They simultaneously and independently decide to stay (s) or raise (r) by betting an additional $b \geq 0$. The high card wins $a + b$, or if one player raises and the other stays, the one raising wins a.

When $n = 3$, if $a \geq b$ a Nash equilibrium involves staying if you pick a 1 and raising otherwise. If $a < b$, a Nash equilibrium is to stay unless you pick the 3. To see this, note that staying if you get the 3 is strongly dominated by raising, so there are four strategies left: rr, rs, sr, and ss, where rr means "raise if you pick a 1, raise if you pick a 2"; rs means "raise if you pick a 1, stay if you pick a 2"; etc. The payoffs, where down the first column, 12 means Alice draws 1, Bob draws 2 etc. are shown in figure 6.9

	rr/rr	rr/rs	rr/sr	rr/ss	rs/rs	rs/sr	rs/ss	sr/sr	sr/ss	ss/ss
12	$-a-b$	a	$-a-b$	a	a	$-a-b$	a	$-a$	0	0
13	$-a-b$	$-a-b$	$-a-b$	$-a-b$	$-a-b$	$-a-b$	$-a-b$	$-a$	$-a$	$-a$
21	$a+b$	$a+b$	a	a	$-a$	0	0	a	a	0
23	$-a-b$	$-a-b$	$-a-b$	$-a-b$	$-a$	$-a$	$-a$	$-a-b$	$-a-b$	$-a$
31	$a+b$	$a+b$	a	a	$a+b$	a	a	a	a	a
32	$a+b$	a	$a+b$	a	a	$a+b$	a	$a+b$	a	a
	0	$2a$	$-2b$	$2(a-b)$	0	$-a-b$	$a-b$	0	$a-b$	0

Figure 6.9. The Groucho Marx game

The conclusion follows directly from the resulting payoff matrix, shown in figure 6.10.

	rr	rs	sr	ss
rr	0	$a/3$	$-b/3$	$(a-b)/3$
rs	$-a/3$	0	$-(a+b)/3$	$(a-b)/3$
sr	$b/3$	$(a+b)/3$	0	$(a-b)/3$
ss	$-(a-b)/3$	$-(a-b)/3$	$-(a-b)/3$	0

Figure 6.10. Payoff matrix for Groucho Marx game

When $n = 4$, we can show that (i) if $b > a = 0$ it is Nash to stay unless you pick the 4; (ii) if $2b > a > 0$, it is Nash to stay unless you get a 3 or a 4; (iii) if $a > 2b > 0$, it is Nash to stay if you get a 1 and raise otherwise.

To see this, note that staying when you pick the 4 is strongly dominated by raising. This leaves us with eight strategies for each player. Staying with 2 and raising with 1 is weakly dominated by staying with 1 or 2. This generalizes to the conclusion that you can eliminate dominated strategies by staying unless the card you pick is greater than some number between 0 and 3. Thus, four strategies remain: {rrr, srr, ssr, sss}. The payoff of any strategy against itself is clearly 0. Thus, it remains to calculate the table in figure 6.11

	rrr/srr	rrr/ssr	rrr/sss	srr/ssr	srr/sss	ssr/sss
12	$-a-b$	a	a	0	0	0
13	$-a-b$	$-a-b$	a	$-a$	0	0
14	$-a-b$	$-a-b$	$-a-b$	$-a$	$-a$	$-a$
21	a	a	a	a	a	0
23	$-a-b$	$-a-b$	a	$-a-b$	a	0
24	$-a-b$	$-a-b$	$-a-b$	$-a-b$	$-a-b$	$-a$
31	a	a	a	a	a	a
32	$a+b$	a	a	a	a	a
34	$-a-b$	$-a-b$	$-a-b$	$-a-b$	$-a-b$	$-a-b$
41	a	a	a	a	a	a
42	$a+b$	a	a	a	a	a
43	$a+b$	$a+b$	a	$a+b$	a	$a+b$
	$-3b$	$2a-4b$	$6a-3b$	$a-2b$	$4a-2b$	$2a$

Figure 6.11. Groucho Marx with $n = 4$

Figure 6.12 shows 12 times the payoff matrix for Alice, from which the conclusion follows.

	rrr	srr	ssr	sss
rrr	0	$-3b$	$2a-4b$	$6a-3b$
srr	$3b$	0	$a-2b$	$4a-2b$
ssr	$-2a+4b$	$-a+2b$	0	$2a$
sss	$6a-3b$	$-4a+2b$	$-2a$	0

Figure 6.12. Payoffs (times 12) for Groucho Marx with $n = 4$

It is possible now to generalize that for any $n > 0$, the only undominated pure strategies take the form of choosing a particular number and raising only if your card is greater than that number. To see this, we represent the strategy of raising if and only if the card chosen is greater than k by s_k. Thus, each player has n pure strategies (eliminating weakly dominated strategies). We must find the payoff to each pure-strategy pair $\{(s_k, s_l)|k, l = 1, \ldots, n\}$. Suppose the pure strategies used are (s_k, s_l) and the cards picked from the hat by Alice and Bob are $\tilde{k}$ and $\tilde{l}$, respectively.

First, suppose $k \geq l$. The probability that Alice wins if both stay is

$$\begin{aligned} \mathrm{P}\left[\tilde{k} > \tilde{l}|\tilde{k} \leq k, \tilde{l} \leq l\right] &= \mathrm{P}\left[\tilde{k} \leq l\right]\mathrm{P}\left[\tilde{k} > \tilde{l}|\tilde{k}, \tilde{l} \leq l\right] \\ &\quad + \mathrm{P}\left[\tilde{k} > l|\tilde{k} \leq k, \tilde{l} \leq l\right] \\ &= \frac{l}{k}\frac{1}{2} + \frac{k-l}{k} = 1 - \frac{l}{2k}. \end{aligned}$$

Because the probability that Alice loses if both stay is one minus the preceding quantity and Alice stands to win or lose a in this case, we find that Alice's expected payoff in this case is

$$\pi_{k \geq l}\left[\tilde{k} > \tilde{l}|\tilde{k} \leq k, \tilde{l} \leq l\right] = a\left(1 - \frac{l}{k}\right).$$

By symmetry (interchange k and l and then negate, or you can calculate it out), we have

$$\pi_{k<l}\left[\tilde{k} > \tilde{l}|\tilde{k} \leq k, \tilde{l} \leq l\right] = -a\left(1 - \frac{k}{l}\right).$$

We also have the following easy payoffs:

$$\begin{aligned} \pi\left[\tilde{k} > k, \tilde{l} \leq l\right] &= a \\ \pi\left[\tilde{k} \leq k, \tilde{l} > l\right] &= -a \end{aligned}$$

Finally, suppose both players raise. First assume $k \geq l$. Then,

$$\begin{aligned} \mathrm{P}\left[\tilde{k} > \tilde{l}|\tilde{k} > k, \tilde{l} > l\right] &= \mathrm{P}\left[\tilde{k} > \tilde{l}|\tilde{k}, \tilde{l} > k\right] + \mathrm{P}\left[\tilde{l} \leq k|\tilde{l} > l\right] \\ &= \frac{n-k}{n-l}\frac{1}{2} + \frac{k-l}{n-l}. \end{aligned}$$

Because the probability that Alice loses if both raise is one minus the preceding quantity and Alice stands to win or lose $a + b$ in this case, we find that Alice's expected payoff in this case is

$$\pi_{k\geq l}\left[\tilde{k} > \tilde{l} | \tilde{k} > k, \tilde{l} > l\right] = (a+b)\frac{k-l}{n-l}.$$

By symmetry (or you can calculate it out), we have

$$\pi_{k<l}\left[\tilde{k} > \tilde{l} | \tilde{k} > k, \tilde{l} > l\right] = (a+b)\frac{k-l}{n-k}.$$

Now we add everything up:

$$\begin{aligned}
\pi_{k\geq l} &= \mathrm{P}\left[\tilde{k} \leq k\right]\mathrm{P}\left[\tilde{l} \leq l\right]\pi_{k\geq l}\left[\tilde{k} > \tilde{l} | \tilde{k} \leq k, \tilde{l} \leq l\right] \\
&\quad + \mathrm{P}\left[\tilde{k} \leq k\right]\mathrm{P}\left[\tilde{l} > l\right]\pi\left[\tilde{k} \leq k, \tilde{l} > l\right] \\
&\quad + \mathrm{P}\left[\tilde{k} > k\right]\mathrm{P}\left[\tilde{l} \leq l\right]\pi\left[\tilde{k} > k, \tilde{l} \leq l\right] \\
&\quad + \mathrm{P}\left[\tilde{k} > k\right]\mathrm{P}\left[\tilde{l} > l\right]\pi_{k\geq l}\left[\tilde{k} > \tilde{l} | \tilde{k} > k, \tilde{l} > l\right] \\
&= \frac{1}{n^2}(l-k)(a(k-l) - b(n-k)).
\end{aligned}$$

By symmetry (or calculation if you do not trust your answer. I did it by calculation and checked it by symmetry), we have

$$\pi_{k<l} = \frac{1}{n^2}(k-l)(a(k-l) + b(n-l)).$$

The reader is invited to write the matrix for the normal form game for $n = 5$, and show that when $a = 1$ and $b = 2$, there is exactly one Nash equilibrium, given by $0.125s_1 + 0.375s_3 + 0.5s_4$. Moreover, for $n = 6$, when $a = 1$ and $b = 2$, there is exactly one Nash equilibrium, given by $0.083s_1 + 0.667s_4 + 0.25s_5$. For $n = 7$, when $a = 1$ and $b = 2$, there is exactly one Nash equilibrium, given by $0.063s_1 + 0.937s_5$.

6.39 Games of Perfect Information

Let $\pi = \{\pi_1, \ldots, \pi_n\}$ be the payoffs in a Nash equilibrium of a finite extensive form game G with perfect information (§5.6). Show that there is a pure-strategy, subgame perfect, Nash equilibrium with payoffs π. Hint: Use mathematical induction on the number of nonterminal nodes in the game.

A Nash equilibrium s is *strict* if there is a neighborhood of s (considered as a point in n-space) that contains no other Nash equilibrium of the game. Strict Nash equilibria of finite games are extremely well behaved dynamically, as we shall see in later chapters. They are especially well behaved if they are unique. A strict Nash equilibrium is always a pure-strategy equilibrium. Give an example of a pure-strategy equilibrium in a game of perfect information that is not strict.

6.40 Correlated Equilibria

	l	r
u	5,1	0,0
d	4,4	1,5

Consider the up-down/left-right game played by Alice and Bob, with normal form matrix shown in the diagram. There are two Pareto-efficient (§5.3) pure-strategy equilibria: (1,5) and (5,1). There is also a mixed-strategy equilibrium with payoffs (2.5,2.5), in which Alice plays u with probability 0.5, and Bob plays l with probability 0.5.

If the players can jointly observe an event with probability 1/2, they can achieve the payoff (3,3) by playing (u,l) when the event occurs, and (d,r) when it does not. Note that this is Nash, because if the event occurs and Bob plays l, Alice's best response is u; if the event does not occur and Bob plays r, then Alice's best response is d; and similarly for Bob. This is called a *correlated equilibrium*.

A more general correlated equilibrium for this coordination game can be constructed as follows. Build a device that has three states: a, b, and c, with probability of occurrence α, β, and $1-\alpha-\beta$. Allow Alice to have the information set $[\{a\},\{b,c\}]$, and allow Bob to have the information set $[\{a,b\},\{c\}]$. For what values of α and β is the following Nash: Alice plays u when she sees a and plays d when she sees $\{b,c\}$; Bob plays r when he sees c and plays l when he sees $\{a,b\}$.

Note that when a occurs, Alice sees a, so she knows that Bob sees $\{a,b\}$, so Bob plays l. Thus, Alice's best response is u. So far, so good. When b occurs, Alice sees $\{b,c\}$, so using Bayes' rule, she knows that Bob sees b with probability $\beta/(1-\alpha)$, and Bob sees c with probability $(1-\alpha-$

$\beta)/(1-\alpha)$. Thus, Alice knows she faces the mixed strategy l played with probability $\beta/(1-\alpha)$ and r played with probability $(1-\alpha-\beta)/(1-\alpha)$. The payoff to u in this case is $5\beta/(1-\alpha)$, and the payoff to d is $4\beta/(1-\alpha)+(1-\alpha-\beta)/(1-\alpha)$. If d is to be a best response, we must thus have $1 \geq \alpha + 2\beta$. If c occurs, the same conditions for Alice hold.

What about the conditions for Bob? When c occurs, Alice sees $\{b,c\}$, so she plays d. Bob's best response is r. So far, so good. When a occurs, Bob sees $\{a,b\}$, so his Bayesian posterior for the probability that Alice sees a is then $\alpha/(\alpha+\beta)$. A straightforward argument, parallel to that of the previous paragraph, shows that playing l is a best response if and only if $\alpha \geq \beta$.

Any α and β that satisfy $1 \geq \alpha + 2\beta$ and $\alpha \geq \beta$ permit a correlated equilibrium. Another characterization is $\beta \leq 1/3$ and $1-2\beta \geq \alpha \geq \beta$. What are the Pareto-efficient choices of α and β? Because the equilibrium is $a \to (u,l)$, $b \to (d,l)$, and $c \to (d,r)$, the payoffs to (a,b,c) are

$$\alpha(5,1) + \beta(4,4) + (1-\alpha-\beta)(1,5) = (1+4\alpha+3\beta, 5-4\alpha-\beta),$$

where $\beta \leq 1/3$ and $1-2\beta \geq \alpha \geq \beta$. This is a linear programming problem. The solution is shown in figure 6.13.

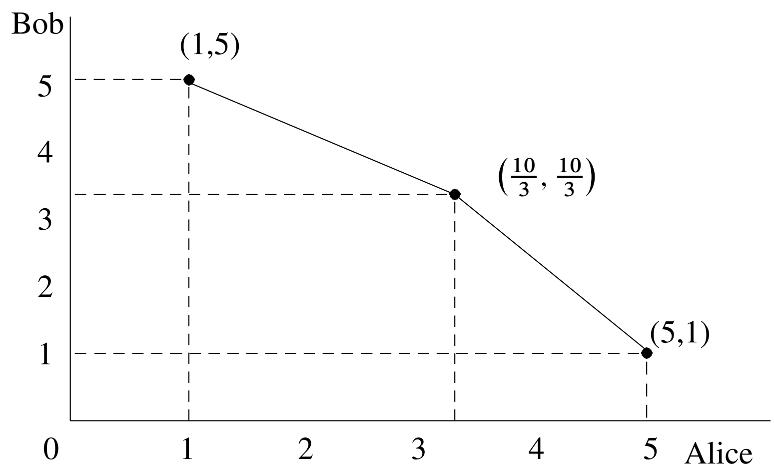

Figure 6.13. Alice and Bob correlate their behavior.

The pair of straight lines connecting (1,5) to (10/3,10/3) to (5,1) is the set of Pareto-efficient points. Note that the symmetric point (10/3,10/3) corresponds to $\alpha = \beta = 1/3$.

6.41 Territoriality as a Correlated Equilibrium

The hawk-dove game (§3.10) is an inefficient way to allocated property rights, especially if the cost of injury w is not much larger than the value v of the property. To see this, note that players choose hawk with probability v/w, and you can check that the ratio of the payoff to the efficient payoff $v/2$ is

$$1 - \frac{v}{w}.$$

When w is near v, this is close to zero.

Suppose some members of the population decide to play a new strategy based on the fact that whenever two players have a property dispute, one of them must have gotten there first, and the other must have come later. We may call the former the "incumbent" and the latter the "contester." The new strategy, B, called the "bourgeois" strategy, always plays hawk when incumbent and dove when contester. When we add B to the normal form matrix of the game, we get the *hawk, dove, bourgeois game* depicted in figure 6.14. Note that the payoff to bourgeois against bourgeois, $v/2$ is greater than $3v/4$–$w/4$, which is the payoff to hawk against bourgeois, and is also greater than $v/4$, which is the payoff to dove against bourgeois. Therefore, bourgeois is a strict Nash equilibrium. It is also efficient, because there is never a hawk-hawk confrontation in the bourgeois equilibrium, so there is never any injury.

	H	D	B
H	$(v-w)/2$	v	$3v/4 - w/4$
D	0	$v/2$	$v/4$
B	$(v-w)/4$	$3v/4$	$v/2$

Figure 6.14. The hawks-dove-bourgeois game

The bourgeois strategy is really a correlated equilibrium of the hawk-dove game, with the correlating device being the signal as to who was the first to occupy the territory. We may think of the signal as a moral justification for ownership.

This example can be widely generalized. Indeed, there are excellent grounds for considering the correlated equilibrium, rather than the Nash equilibrium, the fundamental equilibrium concept in game theory, and for identifying correlated equilibria with social norms (Gintis 2009). Moreover, our species developed through a dynamic call gene-culture coevolution (Boyd and Richerson 1985), and cultural values concerning property rights have been important elements in this coevolutionary process (Gintis 2007).

6.42 Haggling at the Bazaar

Consider seller Alice facing potential buyer Bob in a two-period game. In the first period, Alice makes an offer to sell at price p_1, and Bob accepts or rejects. If Bob accepts, the exchange is made, and the game is over. Otherwise, Alice makes another offer p_2, and Bob accepts or rejects. If he accepts in the second period, the exchange is made. Otherwise, no trade occurs. The game tree is depicted in figure 6.15.

Suppose the reservation price of the good to Alice is s and the value to Bob is b. Suppose Bob and Alice have discount factors δ_a and δ_b for trades that are made in the second period. The value b to Bob is unknown to Alice, but Alice believes that with probability π it is b_h and with probability $1-\pi$ it is b_l, where $b_h > b_l > s$, so Alice would gain from transacting with Bob either way. Suppose that the parameters of the problem are such that if Alice did not get a second chance to make an offer, she would charge the lesser amount b_l.

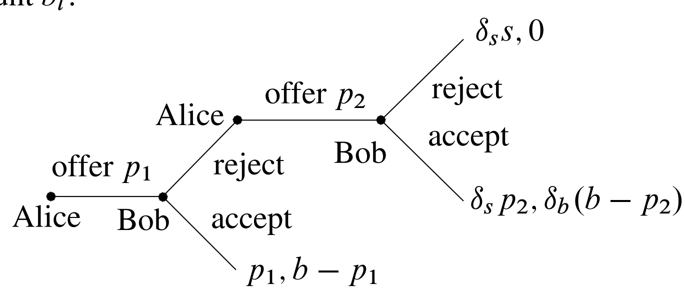

Figure 6.15. Haggling at the bazaar

The payoffs are $(p_1, b-p_1)$, $(\delta_a p_2, \delta_b(b-p_2))$, and $(\delta_a s, 0)$ in case Bob accepts on the first round, the second round, and neither round, respectively. Because, if Alice did not get a second chance to make an offer, she would charge the lesser amount b_l, we must have $\pi b_h + (1-\pi)s \leq b_l$, or simply $\pi \leq (b_l-s)/(b_h-s)$. Suppose $b_h \geq p_1 > b_l$, so there is some chance of getting to the second round. Let $\mu(p_1)$ be Alice's posterior probability that

$b = b_h$, given that Bob refused on round 1 at price p_1. Then, $\mu(p_1) = x\pi/(x\pi + 1-\pi)$ where x is probability that $b = b_h$ and Bob refuses price p_1. To see this, let $\mu = \mathrm{P}(b_h|\text{refuse})$. Then by Bayes' rule, with $x = P(\text{refuse}|b_h)$,

$$\mu = \frac{\mathrm{P}(\text{refuse}|b_h)\mathrm{P}(b_h)}{\mathrm{P}(\text{refuse}|b_h)\mathrm{P}(b_h) + \mathrm{P}(\text{refuse}|b_l)\mathrm{P}(b_l)}$$
$$= \frac{x\pi}{x\pi + 1 - \pi} = \frac{\pi}{\pi + \frac{1-\pi}{x}} \leq \pi.$$

This implies $\mu(p_1) \leq \pi$; that is, if we get to the second round, Alice's posterior probability of the event $\{b = b_h\}$ cannot increase. Thus, we conclude that if we reach a second round, Alice will offer $p_2 = b_l$.

We can now roll back the game tree to the one-stage game in which Alice offers price p_1, the payoff for Bob's strategy "accept the offer" is $(p_1, b{-}p_1)$, and the payoff to his strategy "reject the offer" is $(\delta_a b_l, \delta_b(b{-}b_l))$. Define $p^* = b_h{-}\delta_b(b_h{-}b_l)$. Then, the only undominated strategies for Alice are $p_1 = b_l$ and $p_1 = p^*$. To see this, note that Bob accepts $p_1 = b_l$, so $p_1 < b_l$ is dominated. Any player who accepts p_1 for $b_l < p_1 < p^*$ accepts $p_1 = p^*$. Bob will not accept $p_1 > p^*$ because, whether $b = b_h$ or $b = b_l$, Bob prefers to wait until the second round and get $p_2 = b_l$. At $p_1 = p^*$, the payoffs to "accept" and "reject" are equal if $b = b_h$, because then $b_h{-}p_1 = \delta_b(b_h{-}b_l)$, so Bob accepts on round one if $b = b_h$.

It follows that Alice chooses $p_1 = b_l$ if b_l is greater than $\pi p^* + (1-\pi)\delta_a b_l$, chooses $p_1 = p^*$ if the opposite inequality holds, and otherwise is indifferent between the two choices. The first case reduces to the inequality

$$b_h > \left(\frac{1-\delta_a}{\pi} - (\delta_b - \delta_a)\right)\frac{b_l}{1-\delta_b}. \tag{6.10}$$

We conclude that there is a unique Nash equilibrium.

Now, suppose the parameters of the problem are such that if Alice did not get a second chance to make an offer, she would charge the *greater* amount b_h. Then, there is no Nash equilibrium in pure strategies. To see this, suppose Alice chooses $p_2 = b_l$. Because Alice's posterior probability for $\{b = b_l\}$ cannot be less than π (for the same reason as in the preceding problem) and because she would charge b_h in the one-shot game, she must charge $p_2 = b_h$. So, suppose Alice chooses $p_2 = b_h$. Then, the only undominated strategies on the first round are $p_1 = \delta_b b_h$ and $p_1 = b_l$. But if

Bob rejects $p_1 = \delta_b b_h$, we must have $b = b_l$, so it is not subgame perfect to charge $p_2 = b_h$. The reader is invited to find the mixed-strategy Nash equilibrium of this game.

6.43 Poker with Bluffing Revisited

If you have access to computer software to solve for Nash equilibria of normal form games, you can easily do the following. Doing the problem by hand is not feasible.

a.* Show that there are two Nash equilibria to Poker with Bluffing (§4.16). Ollie uses the same strategy in both, bluffing on the first round with 40% probability (that is, he raises with H or L on the first round, and drops with M). Stan has two mixed-strategy best responses to Ollie, one of which uses two pure strategies and the other uses three. The latter involves bluffing with 40% probability. The expected payoff to the game for Ollie is $1.20. (It's the same for both equilibria, because it's a zero-sum game; see section 6.34).

b.* Now suppose Stan sees Ollie's first move before raising or staying. Show that there are twenty one Nash equilibria, but that Stan uses only two different mixed strategies, and both involve raising with high and raising with medium or low with 25% probability on the first round, and calling with high or medium and calling with 25% probability on the second round. Stan has lots of mixed strategies, but they entail only two different behaviors at the nodes where he chooses. If Ollie raised, Stan raises with H, raises with 75% probability with medium, and stays with low. If Ollie stayed, Stan raises with H, and raises with 25% probability with low. In one set of strategies, Stan raises with 50% probability with medium, and with 25% probability in the other. In all Nash equilibria, Ollie can expect to lose $0.25.

6.44 Algorithms for Finding Nash Equilibria

In all but the simplest cases, finding the complete set of Nash equilibria of a game can be an error-prone chore. If you have the appropriate computer software, however, and if the game is not too complicated, the process can be completely automated. I use Mathematica for this purpose. Even if you do not have access to such software, the analysis in this section is useful, as it supplies considerable insight into the nature of Nash equilibria.

	l	r
A	1,1	1,1
L	$2,-1$	$-10,-5$
R	$-1,-5$	$0,-1$

Mathematica has a command of the form "Solve[*eqns*,*vars*]" that solves a list of equations (*eqns*) for a list of variables (*vars*). For instance, "Solve[{x+y == 5,2x-3y == 7},{x,y}]" would return $x = 22/5$ and $y = 3/5$. Note that in Mathematica, a list is a set of objects, separated by commas, enclosed in a pair of curly brackets, such as {x,y}. Note also that Mathematica uses double equals signs (==) to indicate equality. I will assume the particular game in the diagram. It is easy to see how this generalizes to any finite game. Let $\alpha = \mathrm{P}[A]$, $\beta = \mathrm{P}[L]$, $\gamma = \mathrm{P}[R]$, $\nu = P[l]$, and $\mu = \mathrm{P}[r]$.

The Nash equilibrium payoffs $\bar{\pi}_1$ for player 1 are then given by $\bar{\pi}_1 = \alpha\pi_A + \beta\pi_L + \gamma\pi_R$, where $\gamma = 1 - \alpha - \beta$ and π_A, π_L, and π_R are the payoffs to A, R, and L, respectively. We then have $\pi_A = \nu\pi_{Al} + \mu\pi_{Ar}$, $\pi_L = \nu\pi_{Ll} + \mu\pi_{Lr}$, and $\pi_R = \nu\pi_{Rl} + \mu\pi_{Rr}$, where $\mu = 1 - \nu$ and π_{xy} is the payoff to player 1 choosing pure strategy x and player 2 choosing pure strategy y. Similarly, for player 2, we have $\bar{\pi}_2 = \nu\pi_l + \mu\pi_r$, where $\pi_l = \alpha\pi_{Al} + \beta\pi_{Ll} + \gamma\pi_{Rl}$ and $\pi_r = \alpha\pi_{Ar} + \beta\pi_{Lr} + \gamma\pi_{Rr}$. From the fundamental theorem (§3.6), all the Nash equilibria are solutions to the three equations

$$\alpha(\pi_A - \bar{\pi}_1) = 0, \tag{6.11}$$
$$\beta(\pi_L - \bar{\pi}_1) = 0, \tag{6.12}$$
$$\nu(\pi_l - \bar{\pi}_2) = 0. \tag{6.13}$$

To see this, note first that if the first two equations are satisfied, then $\gamma(\pi_R - \bar{\pi}) = 0$ as well and if the third equation is satisfied, then $\mu(\pi_r - \bar{\pi}_2) = 0$ as well. Suppose $(\bar{\pi}_1, \bar{\pi}_2)$ form a Nash equilibrium. Then if $\pi_x \neq \bar{\pi}_1$ for $x = A, L, R$, we must have $\alpha = 0$, $\beta = 0$, or $\gamma = 0$, respectively. Thus the first two equations in (6.11) hold. A similar argument holds for the third equation.

It is clear, however, that not all the solutions to (6.11) need be Nash equilibria, because these equations do not preclude that pure strategies that do not appear in the Nash equilibrium have higher payoffs than included strategies. To check for this, we may ask Mathematica to list all the payoffs for all the solutions to (6.11), and we can then visually pick out the non-Nash solutions. The Mathematica command for this is

$$\{\{\pi_A, \pi_L, \pi_R, \bar{\pi}_1\}, \{\pi_l, \pi_r, \bar{\pi}_2\}, \{\alpha, \beta, \nu\}\}/.sol$$

Mathematica's response is

$\{\{\pi_A, \pi_L, \pi_R, \bar{\pi}_1\}$	$\{\pi_l, \pi_r, \bar{\pi}_2\}$	$\{\alpha, \beta, \nu\}\}$
$\{1, -10, 0, 0\}$	$\{-5, -1, -1\}$	$\{0, 0, 0\}$
$\{1, 2, -1, -1\}$	$\{-5, -1, -5\}$	$\{0, 0, 1\}$
$\{1, -\frac{10}{13}, -\frac{10}{13}, -\frac{10}{13}\}$	$\{-3, -3, -3\}$	$\{0, \frac{1}{2}, \frac{10}{13}\}$
$\{1, -10, 0, -10\}$	$\{-1, -5, -5\}$	$\{0, 1, 0\}$
$\{1, 2, -1, 2\}$	$\{-1, -5, -1\}$	$\{0, 1, 1\}$
$\{1, 2(6\nu - 5), -\nu, 1\}$	$\{1, 1, 1\}$	$\{1, 0, \nu\}$

The first solution, for $\alpha = \beta = \nu = 0$, is not Nash because 1, the payoff to π_A is greater than 0, the payoff to $\bar{\pi}_1$. The second is not Nash because $1, 2 > -1$, the third is not Nash because $1 > -\frac{10}{13}$, and the fourth is not Nash because $1 > -10$. The fifth is the Nash equilibrium Ll, with payoffs $(2, -1)$. The last is Nash provided $2(6\nu - 5) \leq 1$. In this Nash equilibrium, player 1 plays A and player 2 uses any strategy in which $0 \leq \nu \leq \frac{11}{12}$.

Although this still appears to be somewhat arduous, with the proper software it is almost completely automated, except for the last step, in which non-Nash solutions are discarded. However, with more powerful algorithms, even this step can be automated. To see how, note that if $\bar{\pi} = \sum_i \alpha_i \pi_i$ is the payoff for a player in a Nash equilibrium, in which pure strategy i is played with probability α_i, and π_i is the payoff to strategy i in this equilibrium, then we must have $0 \leq \alpha_i \leq 1$, $\alpha_i(\pi_i - \bar{\pi}) \geq 0$ and $(1 - \alpha_i)(\pi_i - \bar{\pi}) \leq 0$. This is because, by the fundamental theorem (§3.6), if $\alpha_i > 0$, then $\pi_i = \bar{\pi}$, and if $\alpha_i = 0$, then $\pi_i \leq \bar{\pi}$. The converse is also the case: $0 \leq \alpha_i \leq 1$, $\alpha_i(\pi_i - \bar{\pi}) \geq 0$ and $(1 - \alpha_i)(\pi_i - \bar{\pi}) \leq 0$ for all i imply the player is using a best response.

Thus, if player 1 has n pure strategies and player 2 has m pure strategies, we can completely characterize the Nash equilibria by $4(m + n)$ inequalities. Solving these inequalities gives exactly the Nash equilibrium for the problem. Mathematica has an algorithm called "InequalitySolve" that does just this. We must first load this routine with the command <<Algebra‘InequalitySolve‘ . Then, writing

$\gamma = 1 - \alpha - \beta$
$\nu = 1 - \mu$
InequalitySolve[
$\{0 \le \alpha \le 1, 0 \le \beta \le 1, 0 \le \gamma \le 1, 0 \le \nu \le 1, 0 \le \mu \le 1$
$\alpha(\pi_A - \bar{\pi}_1) \ge 0, (1-\alpha)(\pi_A - \bar{\pi}_1) \le 0,$
$\beta(\pi_L - \bar{\pi}_1) \ge 0, (1-\beta)(\pi_L - \bar{\pi}_1) \le 0,$
$\gamma(\pi_R - \bar{\pi}_1) \ge 0, (1-\gamma)(\pi_R - \bar{\pi}_1) \le 0,$
$\nu(\pi_l - \bar{\pi}_2) \ge 0, (1-\nu)(\pi_l - \bar{\pi}_2) \le 0,$
$\mu(\pi_r - \bar{\pi}_2) \ge 0, (1-\mu)(\pi_r - \bar{\pi}_2) \le 0\},$
$\{\alpha, \beta, \nu\}]$

Mathematica returns exactly the set of Nash equilibria:

$$\{\alpha == 0, \beta == 1, \nu == 1\} || \{\alpha == 1, \beta == 0, 0 \le \nu \le \frac{11}{12}\}$$

A note of warning: Mathematica's InequalitySolve does not always find completely mixed Nash equilibria (e.g., try the hawk-dove game), (§3.10), so you should always use both the Solve and InequalitySolve procedures.

We see that Nash equilibria of simple games can be found by a straightforward algorithm. Games that are too complicated for Mathematica to solve can sometimes be solved by hand using mathematical ingenuity. However, creativity and expertise in game theory do not depend on the capacity to solve systems of equations by hand. Rather, creativity and expertise come from understanding how to translate real-life strategic situations into appropriate game-theoretic format, and how to interpret the mechanically derivable results.

6.45 Why Play Mixed Strategies?

In twin sisters (§6.19), we found that there are no pure-strategy Nash equilibria, but there are many mixed-strategy Nash equilibria. In the Nash equilibrium with the highest payoff, each sister asks for \$50 with probability 50/51 and \$51 with probability 1/51. However, if both pure strategies have equal payoffs against the mixed strategy of the other player, why should either sister bother randomizing? Indeed, if one sister conjectures that the other will play her Nash strategy, all mixed strategies have equal payoff, so why prefer the (50/51,1/51) strategy over any other? Moreover, if each sister believes the other is thinking the same way, it is irrational for each to conjecture that the other will choose the (50/51,1/51) strategy or any other particular strategy. Therefore, the whole mixed-strategy Nash equilibrium collapses.

Of course, this problem is not limited to twin sisters; it applies to any strictly mixed Nash equilibrium. By the fundamental theorem (§3.6), any mixed-strategy best response consists of equal-payoff pure strategies, so why should a player bother randomizing? The answer is that there is no reason at all. Therefore, no player should expect any other player to randomize or to do anything else in particular.

You may think that this is some sort of trick argument, or a verbal paradox that can be left to the philosophers to sort out. It is not. The argument completely destroys the classical game-theoretic analysis of mixed-strategy Nash equilibria. In *The Bounds of Reason* (2009), I present some ingenious arguments defending the mixed-strategy equilibrium by Nobel Prize winners John Harsanyi (1973) and Robert Aumann (1987), but their constructions do not work for games that model complex social interaction, such as principal-agent models or repeated games.

Of course, evolutionary game theory has a solution to the problem, provided it is socially meaningful to consider the game itself as the stage game in an evolutionary dynamic in which agents repeatedly meet to play the game and higher-payoff strategies expand as a fraction of the population at the expense of lower-payoff agents. In this setting, each agent plays a pure strategy, but in an evolutionary equilibrium the fraction of each strategy represented in the population equals its weight in the Nash equilibrium. Indeed, this is exactly what we saw when we developed an agent-based model of twin sisters (§6.20).

6.46 Reviewing of Basic Concepts

a. Define a *mixed strategy*, and write the expression for the payoff to using a mixed strategy as a function of the payoffs to the underlying pure strategies.
b. Write the condition for a set of mixed strategies to form a Nash equilibrium.
c. We say a Nash equilibrium is *strict* if the strategy used by each player in this equilibrium is the only best response to the strategies used by the other player. Define a *strictly mixed* strategy, and show that if any player in a Nash equilibrium uses a strictly mixed strategy, then the equilibrium is not strict.