11

Dynamical Systems

History is Spirit at war with itself.
Georg Wilhelm Freidrich Hegel

We have studied Nash equilibria of games, but *do* games reach Nash equilibrium and, if so, by what process? If there are several Nash equilibria, to which one does the game go? Indeed, what are Nash equilibria equilibria of? To answer these questions we will study the behavior of *dynamical systems* that are generally not in equilibrium but which, under appropriate conditions, approach a state of equilibrium over time, or orbit equilibria the way planets orbit the sun, or have some other love-hate relationship with equilibria (e.g., strange attractors).

We can apply several analytical tools in treating strategic interactions as dynamical systems, including difference equations, stochastic processes (such as Markov chains and diffusion processes), statistical mechanics, and differential equations. The differential equation approach is the most basic and has the quickest payoff, so that is what we will develop in this chapter.

11.1 Dynamical Systems: Definition

Suppose $x = (x_1, \ldots, x_n)$ is a point in n-dimensional space $\mathbf{R}^n$ that traces out a curve through time. We can describe this as

$$x = x(t) = (x_1(t), \ldots, x_n(t)) \qquad \text{for } -\infty < t < \infty.$$

Often we do not know $x(t)$ directly, but we do know the forces determining its rate and direction of change in some region of $\mathbf{R}^n$. We thus have

$$\dot{\mathbf{x}} = f(\mathbf{x}) \qquad \mathbf{x} \in \mathbf{R}^n, \tag{11.1}$$

where the "dot" indicates the derivative with respect to t, so $\dot{\mathbf{x}} = dx/dt$. We always assume f has continuous partial derivatives. If we write these vector equations out in full, we get

$$\begin{aligned}\frac{dx_1}{dt} &= f^1(x_1, \ldots, x_n),\\ \frac{dx_2}{dt} &= f^2(x_1, \ldots, x_n),\\ &\vdots\\ \frac{dx_n}{dt} &= f^n(x_1, \ldots, x_n),\end{aligned}$$

We call this a set of *first-order ordinary differential equations* in n unknowns. It is "first-order" because no derivative higher than the first appears. It is "ordinary" as opposed to "partial" because we want to solve for a function of the single variable t, as opposed to solving for a function of several variables.

We call $\mathbf{x}(t)$ a *dynamical system* if it satisfies such a set of ordinary differential equations, in the sense that $\dot{\mathbf{x}}(t) = f(\mathbf{x}(t))$ for t in some (possibly infinite) interval. A *fixed point*, also called a *critical point*, or a *stationary point*, is a point $\mathbf{x}^* \in \mathbf{R}^n$ for which $f(\mathbf{x}^*) = 0$.

11.2 Population Growth

Suppose the rate of growth of fish in a lake is r. Then the number y of fish in the lake is governed by the equation

$$\dot{y} = ry.$$

We can solve this equation by "separation of variables," bringing all the expressions involving t on the right, and all the expressions involving y on the left. This is not possible for just any differential equation, of course, but it is possible in this case. This gives

$$\frac{dy}{y} = r\,dt.$$

Now we integrate both sides, getting $\ln y = rt + a$, where a is a constant of integration. Taking the antilogarithm of both sides, we get

$$y = be^{rt},$$

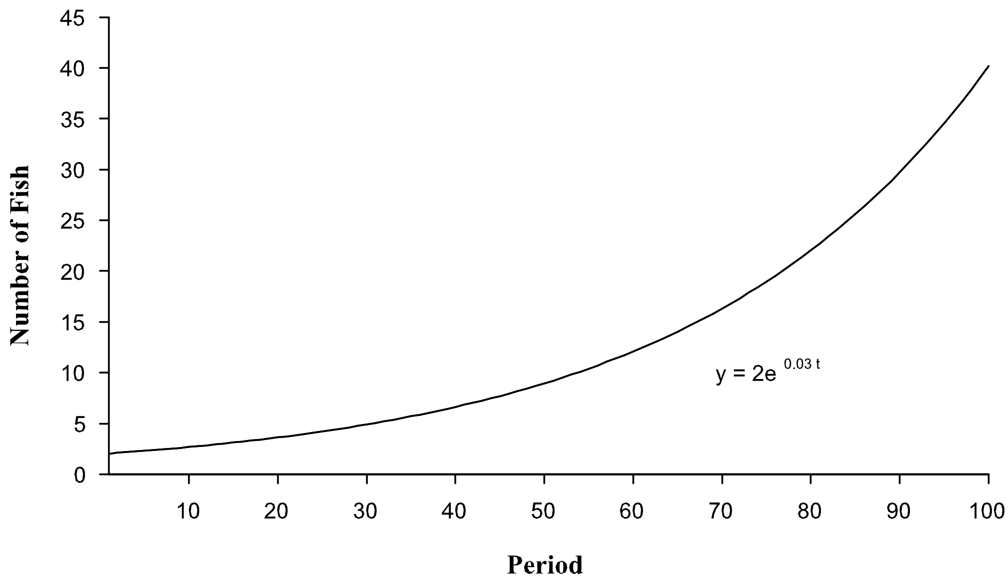

Figure 11.1. The exponential growth of fish in lake. The initial population is $y_o = 2$, and the rate of growth is $r = 3.0\%$.

where $b = e^a$ is another constant of integration.

We determine the constant of integration by noting that if the number of the fish in the lake at time $t = 0$ is y_o, then we must have $b = y_o$. This gives the final solution

$$y = y_o e^{rt}. \tag{11.2}$$

This function is graphed in figure 11.1.

11.3 Population Growth with Limited Carrying Capacity

Equation (11.2) predicts that the fish population can grow without bounds. More realistically, suppose that the more fish, the lower the rate of growth of fish. Let η be the "carrying capacity" of the lake—the number of fish such that the rate of growth of the fish population is zero. The simplest expression for the growth rate of the fish population, given that the growth rate is r when y is near zero, is then $r(1 - y/\eta)$. Our differential equation then becomes

$$\dot{y} = r\left(1 - \frac{y}{\eta}\right) y \qquad \eta, r > 0. \tag{11.3}$$

Note that the dynamical system given by this equation has two fixed points: $y^* = 0$, where the fish population is zero, and $y^* = \eta$, where the population is just equal to the carrying capacity.

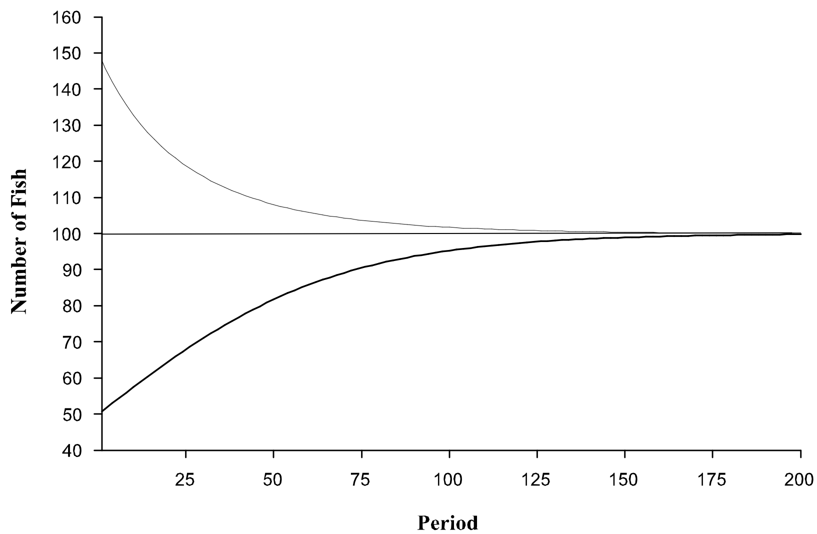

Figure 11.2. Population growth with limited carrying capacity

To solve the equation, we separate variables, getting

$$\frac{dy}{y(\eta - y)} = \frac{r}{\eta} dt.$$

We now integrate both sides, getting

$$\int \frac{dy}{y(\eta - y)} = \frac{r}{\eta} t + a, \tag{11.4}$$

where a is a constant of integration. We use the method of partial fractions to write

$$\frac{1}{y(\eta - y)} = \frac{1}{\eta}\left[\frac{1}{\eta - y} + \frac{1}{y}\right].$$

Thus, we have

$$\int \frac{dy}{y(\eta - y)} = \frac{1}{\eta}\left[\int \frac{dy}{\eta - y} + \int \frac{dy}{y}\right]$$

$$= \frac{1}{\eta} \ln \frac{y}{\eta - y}.$$

Substituting into (11.4), we get

$$\ln \frac{y}{\eta - y} = rt + a\eta.$$

Taking antilogarithms of both sides, this becomes

$$\frac{y}{\eta - y} = be^{rt},$$

where $b = e^{a\eta}$ is another constant of integration. If the number of fish in the lake at time $t = 0$ is y_o, then we must have $b = y_o/(\eta - y_o)$, which can be either positive or negative, depending on whether the initial fish population is larger or smaller than the stationary population size η.

Now we can solve this equation for y, getting

$$y = \frac{\eta}{Ae^{-rt} + 1},$$

where $A = (\eta - y_o)/y_o$. Note that this equation predicts a smooth movement from disequilibrium to stationarity as $t \to \infty$. A picture of the process is given in figure 11.2.

11.4 The Lotka-Volterra Predator-Prey Model

Foxes eat rabbits. Suppose we normalize the rabbit population at a point in time to a fraction x of its maximum, given the carrying capacity of its habitat when foxes are absent, and suppose the fox population at a point in time is a fraction y of its maximum, given the carrying capacity of its habitat when there is an unlimited supply of rabbits. Suppose foxes are born at the rate $\delta_1 x$ but die at the rate $\gamma_1(1-x)$. We then have $\dot{y}/y = \delta_1 x - \gamma_1(1-x)$, which we can write as

$$\dot{y} = \delta y(x - \gamma), \qquad \delta > 0, 1 > \gamma > 0, \tag{11.5}$$

where we have written $\delta = \delta_1 + \gamma_1$ and $\gamma = \gamma_1/(\delta_1 + \gamma_1)$. Equation (11.5) expresses the rate of growth $\dot{y}/y$ as a function of the frequency of rabbits.

Suppose the natural rate of growth of rabbits is $g > 0$, but predation reduces the rate of growth by μy, so

$$\dot{x} = x(g - \mu y). \tag{11.6}$$

Now, (11.5) and (11.6) form a pair of differential equations in two unknowns (x and y), the solution to which is a dynamical system known as the *Lotka-Volterra predator-prey model*.

How do we solve this equation? There is no solution in closed form (e.g., using polynomials, trigonometric functions, logarithms, and exponentials). We can, however, discover the properties of such equations without solving them explicitly.

We begin such an analysis with a *phase diagram* of the differential equations. The phase diagram for the Lotka-Volterra model is depicted in figure 11.3.

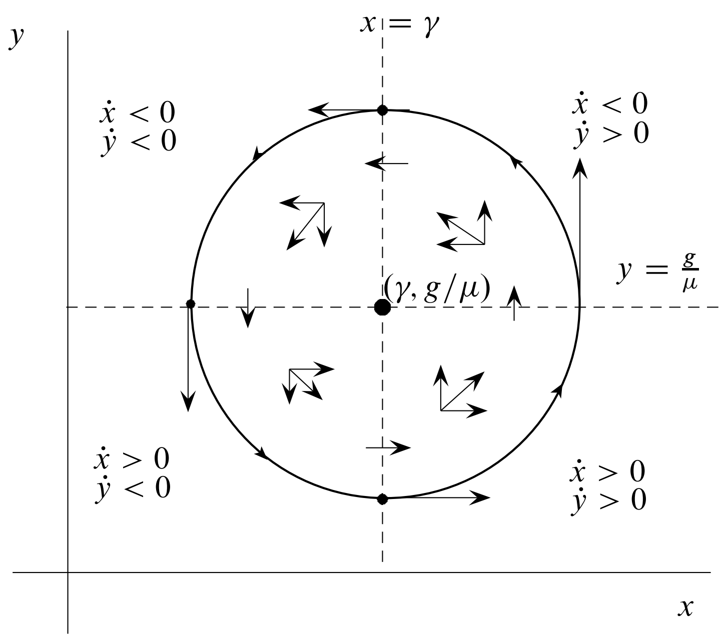

Figure 11.3. Phase diagram of Lotka-Volterra system

The horizontal dotted line represents the condition $dx/dt = 0$, and the vertical dotted line represents the condition $dy/dt = 0$. The fixed point is at $(\gamma, g/\mu)$, where the two intersect. The little arrows show in which direction the flow of the dynamical system moves for that particular point

(x, y). The arrows point northward when $dy/dt > 0$ and southward when $dy/dt < 0$, and they point eastward when $dx/dt > 0$ and westward when $dx/dt < 0$. The arrows are vertical where $dx/dt = 0$ because motion is purely north-south instantaneously at such a point, and are horizontal where $dy/dt = 0$, because motion is purely east-west at such a point. In each of the four quadrants marked off by the dotted lines, the direction of the flow is qualitatively similar. Thus to the northeast of the fixed point, the flow is northwest; to the northwest, the flow is southwest; to the southwest, the flow is southeast; and to the southeast of the fixed point, the flow is to the northeast. So it is clear that the flow circles counterclockwise about the fixed point. However, we cannot tell *a priori* whether the flow circles into the fixed point, circles outward to infinity, or forms closed circuits about the fixed point.

To show that the Lotka-Volterra has closed orbits (§11.5), we find a function that is constant on any trajectory of the dynamical system and show that this function is monotonic (strictly increasing or decreasing) along a ray starting from the fixed point and pointing northeast.[1]

Suppose we have such a function f and consider a path starting at a point $\mathbf{x}$ on the ray, making one complete revolution around the fixed point and hitting the ray again, say at $\mathbf{y}$. Because f is constant on the path, $f(\mathbf{x}) = f(\mathbf{y})$. But because f is monotonic on the ray, we must have $\mathbf{x} = \mathbf{y}$, so the path is a closed orbit (§11.5). First, we eliminate t from the equations for $\dot{x}$ and $\dot{y}$ by dividing the first by the second, getting

$$\frac{dy}{dx} = \frac{\delta y(x - \gamma)}{x(g - \mu y)}.$$

Now we separate variables, pulling all the x's to the right, and all the y's to the left:

$$\frac{g - \mu y}{y} dy = \frac{\delta(x - \gamma)}{x} dx.$$

Now we integrate both sides, getting

$$g \ln y - \mu y = \delta x - \delta\gamma \ln x + C,$$

[1] We say a function $f(x)$ is (1) *increasing* if $x > y$ implies $f(x) \geq f(y)$; (2) *strictly increasing* if $x > y$ implies $f(x) > f(y)$; (3) *decreasing* if $x > y$ implies $f(x) \leq f(y)$; and (4) *strictly decreasing* if $x > y$ implies $f(x) < f(y)$.

where C is an arbitrary constant of integration. Bringing all the variables over to the left and taking the antilogarithm, we get

$$y^g x^{\delta\gamma} e^{-(\mu y+\delta x)} = e^C. \tag{11.7}$$

So now we have an expression that is constant along any trajectory of the Lotka-Volterra dynamical system.

Now, consider a ray (x, y) that starts at the fixed point $(\gamma, g/\mu)$ and moves to the northeast in a direction heading away from the origin. We can write this as $x = \gamma s$, $y = (g/\mu)s$, where s is a parameter measuring the distance from the fixed point. Note that when $s = 1$, (x, y) is at the fixed point. Substituting in (11.7), we get

$$\left(\frac{g}{\mu}\right)^g \gamma^{\delta\gamma} s^{g+\delta\gamma} e^{-(g+\delta\gamma)s} = e^C.$$

This looks forbidding, but it's really not. We pull the first two terms on the left over to the right, and then take the $(g + \delta\gamma)$-th root of both sides. The right-hand side is a complicated constant, which we can abbreviate by D, and the left is just se^{-s}, so we have

$$se^{-s} = D. \tag{11.8}$$

If we can show that the left-hand side is strictly decreasing for $s > 1$, we are done, because then any $s > 1$ that satisfies (11.8) must be unique. We take the derivative of the left-hand side, getting

$$e^{-s} - se^{-s} = (1 - s)e^{-s},$$

which is negative for $s > 1$. This shows that the dynamical system moves in closed orbits (§11.5) around the fixed point.

It follows from this analysis that if the system begins out of equilibrium, both the fraction of rabbits and foxes will go through constant-amplitude oscillations around their equilibrium values forever. We shall later characterize this as an *asymmetric evolutionary game* (§12.17) for which this oscillatory behavior is quite typical.

11.5 Dynamical Systems Theory

With these examples under our belt, we can address the basic theory of dynamical systems (a.k.a. differential equations).[2]

Suppose a dynamical system is at a point $\mathbf{x}_o$ at time t_o. We call the locus of points through which the system passes as $t \to \infty$ the *forward trajectory* of the system through $\mathbf{x}_o$, or the *trajectory* of the system starting at $\mathbf{x}_o$. The *backward trajectory* of the system through $\mathbf{x}_o$ is the locus of points through which the system passes as $t \to -\infty$. The forward and backward trajectories are together called the *trajectory* through $\mathbf{x}_o$.

Clearly if a dynamical system is at a fixed point $\mathbf{x}^*$, it will stay there forever, so the trajectory starting at $\mathbf{x}^*$ is simply $\mathbf{x}^*$ itself. However, if we perturb the system a little from $\mathbf{x}^*$ by choosing a new initial point $\mathbf{x}_o$ at time $t = 0$, there are several things that can happen. We begin with a couple of definitions. If $\mathbf{x} \in \mathbf{R}^n$, and $r > 0$, we define a *ball of radius r* around $\mathbf{x}$, which we write $B_r(\mathbf{x})$, as the set of points $\mathbf{y} \in \mathbf{R}^n$ whose distance from $\mathbf{x}$ is less than r. We define a *neighborhood* of $\mathbf{x}$ to be any subset of $\mathbf{R}^n$ that contains some ball around $\mathbf{x}$. Finally, we say a set in $\mathbf{R}^n$ is an *open set* if it is a neighborhood of each of its points. Note that a set is open if and only if it contains a ball of some positive radius around each of its points.

We define an *ϵ-perturbation* of the dynamical system at a fixed point $\mathbf{x}^*$ to be a trajectory of the system starting at some $\mathbf{x}_o \in B_\epsilon(\mathbf{x}^*)$, where $\epsilon > 0$ and $\mathbf{x}_o \neq \mathbf{x}^*$. We say a trajectory $\mathbf{x}(t)$ *approaches* $\mathbf{x}^*$ if $\mathbf{x}(t) \to \mathbf{x}^*$ as $t \to \infty$. We say a trajectory $\mathbf{x}(t)$ *ϵ-escapes* $\mathbf{x}^*$ if there is some t_o such that $\mathbf{x}(t) \notin B_\epsilon(\mathbf{x}^*)$ for $t > t_o$; that is, after some point in time, the trajectory never gets closer than ϵ to $\mathbf{x}^*$.

If there is some $\epsilon > 0$ such that for any $\mathbf{x}_o \in B_\epsilon(\mathbf{x}^*)$, the trajectory through $\mathbf{x}_o$ approaches $\mathbf{x}^*$, we say the fixed point at $\mathbf{x}^*$ is *asymptotically stable*. The set of points $\mathbf{x}_o \in \mathbf{R}^n$ such that a trajectory through $\mathbf{x}_o$ approaches $\mathbf{x}^*$ is called the *basin of attraction* of the fixed point $\mathbf{x}^*$. If every point where the differential equation is defined is in the basin of attraction of $\mathbf{x}^*$, we say the fixed point is *globally stable*.

If $\mathbf{x}^*$ is not asymptotically stable, but for any ball $B_\epsilon(\mathbf{x}^*)$ there is another ball $B_\delta(\mathbf{x}^*)$ such that for any point $\mathbf{x}_o \in B_\delta(\mathbf{x}^*)$, the trajectory starting at $\mathbf{x}_o$

[2] There are many excellent texts on differential equations. Some of my favorites are Perko 1991, Hirsch and Smale 1974, Epstein 1997, and Hofbauer and Sigmund 1998. The last of these is a beautiful summary of evolutionary dynamics.

never leaves $B_\epsilon(\mathbf{x}^*)$, we say the fixed point at $\mathbf{x}^*$ is *neutrally stable*. Neutral stability means that a sufficiently small perturbation about the fixed point never leads the system too far away from the fixed point. A special case is when any trajectory through $\mathbf{x}_o \in B_\epsilon(\mathbf{x}^*)$ is a *closed orbit*; that is, the trajectory starting at $\mathbf{x}_o$ eventually returns to $\mathbf{x}_o$.

If $\mathbf{x}^*$ is neither asymptotically stable nor neutrally stable, we say $\mathbf{x}^*$ is *unstable*. Thus, $\mathbf{x}^*$ is unstable if there is an $\epsilon > 0$ such that for any ball $B_\delta(\mathbf{x}^*)$, there is a point $\mathbf{x}_o \in B_\delta(\mathbf{x}^*)$ such that the trajectory starting at $\mathbf{x}_o$ ϵ-escapes $\mathbf{x}^*$.

11.6 Existence and Uniqueness

THEOREM 11.1 Existence, Uniqueness, and Continuous Dependence on Initial Conditions. *Suppose that f in equation (11.1) has continuous derivatives on an open set D containing a point x_o. Then there is some interval $I = [-t_o, t_o]$ and a unique trajectory $\mathbf{x}(t)$ satisfying (11.1) defined on I with $\mathbf{x}(0) = x_o$. Moreover, $\mathbf{x}(t)$ depends smoothly upon x_o in the following sense: there is some $\delta > 0$, and a unique function $\mathbf{x}(t, \mathbf{y})$ that satisfies (11.1) on an interval $[-t_1, t_1]$ with $\mathbf{x}(0, \mathbf{y}) = \mathbf{y}$, for all $\mathbf{y} \in B_\delta(\mathbf{x}_o)$. Moreover, $\mathbf{x}(t, \mathbf{y})$ has continuous partial derivatives, and continuous second partial derivatives with respect to t.*

This theorem says that if $f(\mathbf{x})$ is suitably well behaved, the dynamical system (11.1) has a unique, twice-differentiable trajectory through each point $\mathbf{x}_o$, and the trajectory varies differentiably as we vary $\mathbf{x}_o$. In particular, two trajectories can never cross.

THEOREM 11.2 Continuous Dependence on Parameters. *Let $\mu \in \mathbf{R}^k$ be a set of k parameters, and suppose $f(\mathbf{x}, \mu)$ has continuous partial derivatives in a neighborhood of $(\mathbf{x}_o, \mu_o) \in \mathbf{R}^{n+k}$. Then there is a $t_1 > 0$, a $\delta > 0$, an $\epsilon > 0$, and a unique function $\mathbf{x}(t, \mathbf{y}, \mu)$ that satisfies*

$$\dot{\mathbf{x}} = f(\mathbf{x}(t, \mathbf{y}, \mu), \mu) \tag{11.9}$$

with $\mathbf{x}(0, \mathbf{y}, \mu) = \mathbf{y}$, for $t \in [-t_1, t_1]$, $\mathbf{y} \in B_\delta(\mathbf{x}_o)$, and $\mu \in B_\epsilon(\mu_o)$. Moreover, $\mathbf{x}(t, \mathbf{y}, \mu)$ has continuous partial derivatives.

This theorem says that if $f(\mathbf{x}, \mu)$ is suitably well behaved, the trajectories of the dynamical system (11.9) vary differentiably as we vary the parameters μ.

11.7 The Linearization Theorem

Given a dynamical system (11.1), we define the *Jacobian* of f at a point $\mathbf{x} \in \mathbf{R}^n$ to be the $n \times n$ matrix $Df(\mathbf{x}) = (a_{ij})$ where

$$a_{ij} = \frac{\partial f^i}{\partial x_j}(\mathbf{x}) \qquad \text{for } i, j = 1, \ldots, n.$$

Suppose $\mathbf{x}^*$ is a fixed point of the dynamical system (11.1), and let $A = Df(\mathbf{x}^*)$ be the Jacobian of the system at $\mathbf{x}^*$. We define the *linearization* of the original dynamic system (11.1) at $\mathbf{x}^*$ to be the linear dynamical system

$$\begin{aligned}
\dot{x}_1 &= a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n,\\
\dot{x}_2 &= a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n,\\
&\vdots\\
\dot{x}_n &= a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n,
\end{aligned}$$

or, more succinctly,

$$\dot{\mathbf{x}} = A\mathbf{x} \qquad \mathbf{x} \in \mathbf{R}^n. \tag{11.10}$$

Note that the fixed point of linearization has been moved from $\mathbf{x}^*$ to 0 (we could keep the fixed point at $\mathbf{x}^*$ by defining the linearization as $\dot{\mathbf{x}} = A(\mathbf{x} - \mathbf{x}^*)$, but this needlessly complicates the notation).

We define the eigenvalues of the matrix A in (11.10) to be the set of (possibly complex) numbers λ that satisfy the equation

$$A\mathbf{x} = \lambda\mathbf{x} \tag{11.11}$$

for some vector $\mathbf{x} \neq 0$. This equation can be rewritten as $(A - \lambda I)x = 0$, which holds for $x \neq 0$ only if the determinant of $A - \lambda I$ is zero. This determinant is given by

$$|A - \lambda I| = \begin{vmatrix} a_{11} - \lambda & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} - \lambda & \ldots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} - \lambda \end{vmatrix}$$

Because this is a polynomial of degree n, we know from linear algebra—you can refer to Hirsch and Smale (1974) for details—there are exactly n (possibly complex) eigenvalues, if we account properly for their "multiplicity." At any rate, we shall only deal in this book with dynamical systems in one or two dimensions, where the calculation of the eigenvalues is very simple.

We call the dynamical system (11.1) *hyperbolic* at a fixed point $\mathbf{x}^*$ if every eigenvalue of the Jacobian matrix $Df(x^*)$ has nonzero real part. We then have the following.

THEOREM 11.3 Linearization Theorem. *Suppose the dynamical system (11.1) is hyperbolic at fixed point* $\mathbf{x}^*$. *Then* $\mathbf{x}^*$ *is asymptotically stable if its linearization (11.10) is asymptotically stable. Also, if* $\mathbf{x}^*$ *is asymptotically stable, then no eigenvalue of the Jacobian matrix* $Df(x^*)$ *has strictly positive real part.*

When no eigenvalue has a strictly positive real part at $\mathbf{x}^*$, but one or more eigenvalues have a zero real part, $\mathbf{x}^*$ may be either stable or unstable.

11.8 Dynamical Systems in One Dimension

If $n = 1$, equation (11.1) becomes

$$\dot{x} = f(x), \qquad x \in \mathbf{R}. \tag{11.12}$$

Suppose $f(x)$ has the shape shown in figure 11.4. We call a diagram like the one in figure 11.4 a *phase diagram*—a depiction of the state space of the dynamic system with little arrows showing the direction of movement of the system at representative points in the state space.

It is obvious that fixed points x_a and x_c are stable, whereas fixed point x_b is unstable. To see that this agrees with the linearization theorem 11.3, note that the Jacobian at a point x is just the one-dimensional matrix $(f'(x))$, and the eigenvalue of this matrix is just $f'(x)$. Thus, the system has a fixed point at x^* if $f(x^*) = 0$, and this fixed point is hyperbolic if $f'(x^*) \neq 0$. Note that in figure 11.4 all three fixed points are hyperbolic, because $f'(x) < 0$ at x_a and x_c, and $f'(x) > 0$ at x_b. The linearization of (11.12) at fixed point x^* is $\dot{x} = f'(x^*)x$, which has solution

$$x(t) = x(0)e^{f'(x^*)t},$$

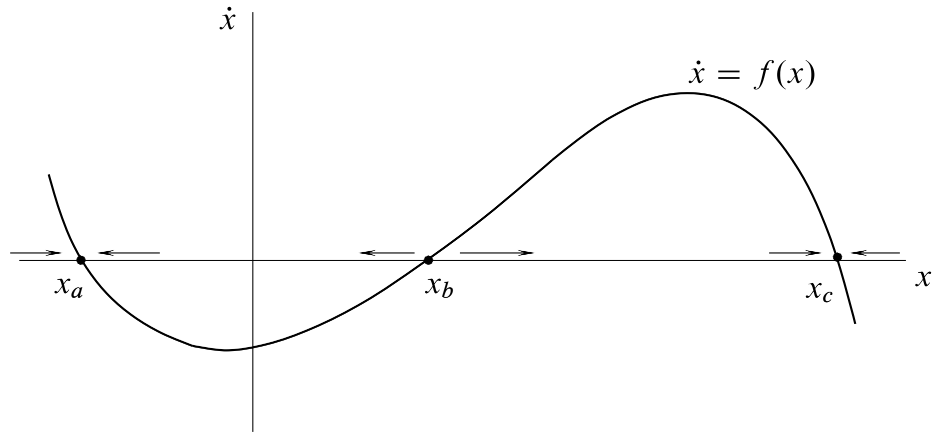

Figure 11.4. Stable and unstable fixed points

which is obviously stable when $f'(x^*) < 0$ and unstable when $f'(x^*) > 0$. Applying the linearization theorem, we find that the fixed points at x_a and x_c are stable, whereas fixed point x_b is unstable.

We can also apply the linearization theorem to the population growth with limited carrying capacity dynamical system (11.3). This system has two fixed points, $y = 0$ and $y = \eta$. The Jacobian at a point y is just

$$r\left(1 - \frac{2y}{\eta}\right),$$

which has the value $r > 0$ at $y = 0$ and the value $-r < 0$ at $y = \eta$. The linearization of the dynamical system at $y = 0$ is thus

$$\dot{y} = ry,$$

which has the solution $y = ae^{rt}$. This explodes to infinity, so $y = 0$ is unstable.

The linearization of the dynamical system at $y = \eta$ is

$$\dot{y} = -ry$$

with solution $y = ae^{-rt}$. This converges to zero so $y = \eta$ is an asymptotically stable fixed point.

We conclude that in this model, the fixed point $y = \eta$ is globally stable, and is approached exponentially from any $y > 0$.

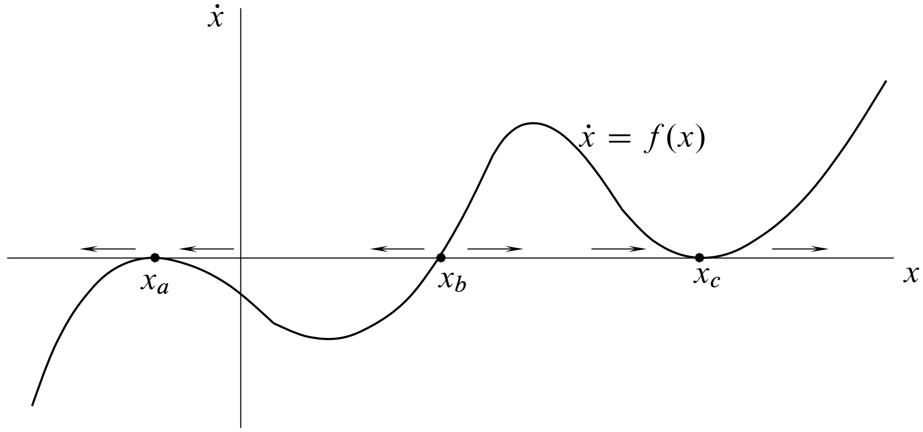

Figure 11.5. The phase diagram of nonhyperbolic fixed points in a one-dimensional system. Note that x_a is unstable to the left and locally stable to the right, B is unstable, and C is locally stable to the left, unstable to the right.

If (11.12) is not hyperbolic at a fixed point x^* (that is, $f'(x^*) = 0$), then we cannot apply the linearization theorem. We illustrate this in the phase diagram shown in figure 11.5. Here the fixed point at x_b is unstable, just as before. But at fixed points x_a and x_c, the Jacobians are zero (that is, $f'(x_a) = f'(x_c) = 0$), so the fixed points are not hyperbolic. Note that linearization has the solution $x(t) = 0$, which of course tells us nothing about the dynamical system. In fact, we can easily see that the system approaches the fixed point from the right of x_a but ϵ-escapes the fixed point to the left of x_a for small ϵ. At x_c the system approaches the fixed point from the left of x_c but ϵ-escapes the fixed point from right of x_c for small ϵ.

11.9 Dynamical Systems in Two Dimensions

We can write the equations for a dynamical system in two dimensions as

$$\begin{aligned} \dot{x} &= f(x, y), \\ \dot{y} &= g(x, y). \end{aligned} \tag{11.13}$$

Suppose this has a fixed point at a point (x^*, y^*). We can write the Jacobian of the system at this point as

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} f_x(x^*, y^*) & f_y(x^*, y^*) \\ g_x(x^*, y^*) & g_y(x^*, y^*) \end{bmatrix}.$$

The linearization of the dynamical system about (x^*, y^*) can then be written as

$$\dot{\mathbf{x}} = A\mathbf{x}, \tag{11.14}$$

where $\mathbf{x}$ is the column vector $\begin{bmatrix} x \\ y \end{bmatrix}$. Let α = trace of $A/2 = (a_{11} + a_{22})/2$, let $\beta = \det(A) = a_{11}a_{22} - a_{21}a_{12}$, and let $\gamma = \alpha^2 - \beta$, the *discriminant* of A. It is easy to check that the eigenvalues of (11.14) are $\lambda_1 = \alpha + \sqrt{\gamma}$ and $\lambda_2 = \alpha - \sqrt{\gamma}$. Note that if $\beta = 0$, the two equations in (11.14) are multiples of each other, so the system is indeterminate. Thus, we assume that $\beta \neq 0$, which implies that (11.14) has the unique critical point $(0, 0)$. We have the following.

THEOREM 11.4 *If $\gamma > 0$, the dynamical system (11.14) is governed by the equations*

$$x(t) = ae^{\lambda_1 t} + be^{\lambda_2 t} \tag{11.15}$$

$$y(t) = ce^{\lambda_1 t} + de^{\lambda_2 t} \tag{11.16}$$

for constants a, b, c, and d that depend on the initial conditions. It follows that the dynamical system is hyperbolic with distinct eigenvalues $\lambda_1 = \alpha + \sqrt{\gamma}$ and $\lambda_2 = \alpha - \sqrt{\gamma}$.

a. *If $\lambda_1, \lambda_2 < 0$, which occurs when $\alpha < 0$ and $\beta > 0$, the fixed point at (0,0) is globally stable. This is called a* stable node.
b. *If $\lambda_1, \lambda_2 > 0$, which occurs when $\alpha, \beta > 0$, the fixed point at (0,0) is unstable and every trajectory starting at a nonfixed point (x_o, y_o) approaches ∞ as $t \to \infty$. This is called an* unstable node .
c. *If $\lambda_1 > 0$ and $\lambda_2 < 0$, which occurs when $\beta < 0$, the system is unstable, but if (x_o, y_o) lies on the straight line*

$$(a_{11} - \lambda_2)x + a_{12}y = 0,$$

the system converges to the fixed point as $t \to \infty$. This line is called the stable manifold *of the system. Also, if (x_o, y_o) lies on the straight line*

$$(a_{11} - \lambda_1)x + a_{12}y = 0,$$

the system converges to the fixed point as $t \to -\infty$. *This line is called the* unstable manifold *of the system. The fixed point is called a* saddle point.

d. If the system starts at $(x(0), y(0)) = (x_o, y_o)$*, then*

$$a = \frac{(a_{11} - \lambda_2)x_o + a_{12}y_o}{2\sqrt{\gamma}}, b = -\frac{(a_{11} - \lambda_1)x_o + a_{12}y_o}{2\sqrt{\gamma}} \qquad (11.17)$$
$$c = \frac{a_{21}x_o + (a_{22} - \lambda_2)y_o}{2\sqrt{\gamma}}, d = -\frac{a_{21}x_o + (a_{22} - \lambda_1)y_o}{2\sqrt{\gamma}}.$$

The proof of this theorem, which is left to the reader, is simple. The main point is to show that the answer satisfies (11.14). By theorem 11.1, there are no other solutions. The constants a, b, c, and d in (11.17) are solutions to the four equations

$$\begin{aligned} x(0) &= x_o = a + b \\ y(0) &= y_o = c + d \\ \dot{x}(0) &= a_{11}x_o + a_{12}y_o = a\lambda_1 + b\lambda_2 \\ \dot{y}(0) &= a_{21}x_o + a_{22}y_o = c\lambda_1 + d\lambda_2, \end{aligned}$$

which follow directly from (11.14), (11.15), and (11.16).

The phase diagram for a stable node in the case $\gamma > 0$ is elementary: the trajectories all converge to the fixed point; in the case of an unstable node, the trajectories all move away from the fixed point. But the case of a saddle point is more interesting and is depicted in figure 11.6.

THEOREM 11.5 *If* $\gamma < 0$*, the dynamical system (11.14) is satisfied by the equations*

$$x(t) = e^{\alpha t}[a \cos \omega t + b \sin \omega t] \qquad (11.18)$$
$$y(t) = e^{\alpha t}[c \cos \omega t + d \sin \omega t], \qquad (11.19)$$

where $\omega = \sqrt{-\gamma}$. *The system is hyperbolic if and only if* $\alpha \neq 0$*, and its eigenvalues are* $\lambda = \alpha \pm \omega\sqrt{-1}$*, so trajectories circle around the fixed point with period* $2\pi/\omega$.

a. If $\alpha < 0$*, the fixed point is globally stable. This is called a* stable focus.
b. If $\alpha > 0$*, the fixed point is unstable. This is called an* unstable focus.

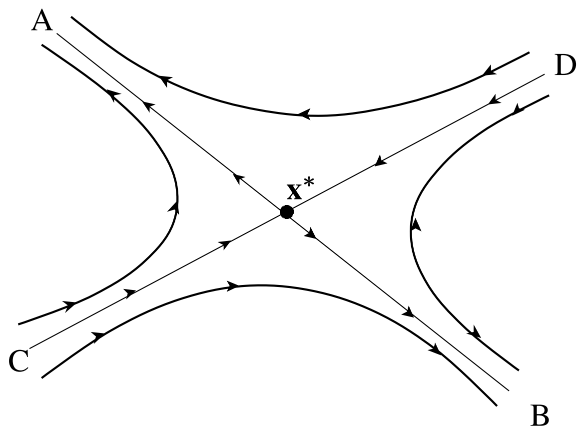

Figure 11.6. The stable and unstable manifolds of a saddle point. Note that AB is the unstable manifold and CD is the stable manifold of the saddle point.

c. *If $\alpha = 0$, the fixed point is neutrally stable, and all trajectories are closed orbits. This is called a* center.

d. *If the system starts at $(x(0), y(0)) = (x_o, y_o)$, then*

$$
\begin{aligned}
a &= x_o, \quad b = \frac{(a_{11} - a_{22})x_o + 2a_{12}y_o}{2\omega} \\
c &= y_o, \quad d = \frac{2a_{21}x_o + (a_{22} - a_{11})y_o}{2\omega}.
\end{aligned}
\tag{11.20}
$$

Note that the coefficients in (11.20) are derived from the solutions to (11.18) and (11.19) for $t = 0$. To understand why the trajectories circle the critical point $(0, 0)$, note that from elementary trigonometry, we have

$$
\begin{aligned}
a\cos\omega t + b\sin\omega t &= a_o\left(a'\cos\omega t + b'\sin\omega t\right) \\
&= a_o(\cos\theta\cos\omega t + \sin\theta\sin\omega t) \\
&= a_o\cos(\omega t - \theta)
\end{aligned}
$$

where $a_o = \sqrt{a^2 + b^2}$, $a' = a/a_o$, $b' = b/a_o$, and $\theta = \arccos a'$. A similar equation holds for both $\dot{x}$ and $\dot{y}$, so the trajectory of (11.20) is an ellipse.

THEOREM 11.6 *If $\gamma = 0$, the dynamical system (11.14) satisfies the equations*

$$x(t) = e^{\alpha t}(at + b) \tag{11.21}$$

$$y(t) = e^{\alpha t}(ct + d), \tag{11.22}$$

and if the system starts at $(x(0), y(0)) = (x_o, y_o)$*, we have*

$$a = (a_{11} - \alpha)x_o + a_{12}y_o \quad b = x_o \tag{11.23}$$

$$c = a_{21}x_o + (a_{22} - \alpha)y_o \quad d = y_o. \tag{11.24}$$

The system has the single eigenvalue α*, and it is hyperbolic if and only if* $\alpha \neq 0$.

a. If $\alpha > 0$*, the origin is an unstable node.*
b. If $\alpha < 0$*, the origin is an stable node.*

figure 11.7 summarizes the behavior of the linear two-dimensional system of differential equations. Note that we have not said what happens when $\beta = \det(A) = 0$. This is called a *degenerate critical point* and is not of much interest. It means one of the differential equations is a multiple of the other, so there is really only one equation.

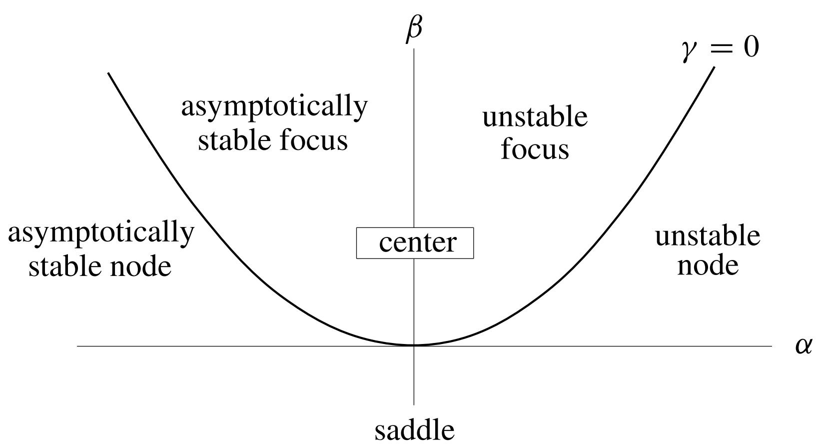

Figure 11.7. A summary of types of fixed points in a two-dimensional dynamical system. The discriminant γ is negative within the parabola $\gamma = 0$ and positive without.

11.10 Exercises in Two-Dimensional Linear Systems

For each of the following differential equations, draw a phase diagram, write out the general solution, and use the results of section 11.9 to determine analytically the nature of the fixed point. Determine the path through the point $(x(0), y(0)) = (1, 1)$.

a. $\dot{x} = \lambda x$, $\dot{y} = \mu y$ for $\lambda, \mu > 0$.
b. $\dot{x} = \lambda x$, $\dot{y} = \mu y$ for $\lambda > 0$, $\mu < 0$.
c. $\dot{x} = \lambda x$, $\dot{y} = \mu y$ for $\lambda, \mu < 0$.
d. $\dot{x} = \lambda x + y$, $\dot{y} = \lambda y$ for $\lambda > 0$.
e. $\dot{x} = \lambda x + y$, $\dot{y} = \lambda y$ for $\lambda < 0$.
f. $\dot{x} = ax - by$, $\dot{y} = bx + ay$ for $a, b > 0$.
g. $\dot{x} = -x - y$, $\dot{y} = x - y$.
h. $\dot{x} = 3x - 2y$, $\dot{y} = x + y$.
i. $\dot{x} = 3x + y$, $\dot{y} = -x + y$.
j. $\dot{x} = y$, $\dot{y} = -x + 2y$.

For instance, the phase diagram for problem f is shown in figure 11.8. It is clear that this is a focus, but we cannot tell whether it is stable or unstable. Indeed, this should depend on the the parameters a and b. The matrix of the dynamical system is given by

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix},$$

so we have $\alpha = a$, $\beta = a^2 + b^2$, $\gamma = -b^2$, and the eigenvalues are $\lambda_1 = a + ib$ and $\lambda_2 = a - ib$. Because $a > 0$, this is an unstable focus.

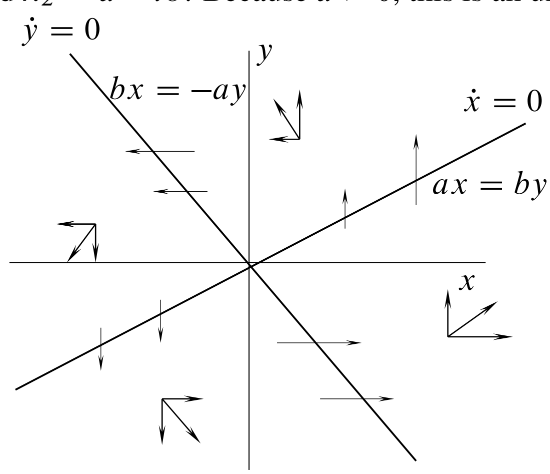

Figure 11.8. Phase Diagram for Problem f

11.11 Lotka-Volterra with Limited Carrying Capacity

The general Lotka-Volterra model (§11.4) has the form

$$\begin{aligned} \dot{x} &= x(a - by), \\ \dot{y} &= y(-c + dx). \end{aligned} \tag{11.25}$$

This model has the absurd property that if the predator is absent, the prey grows at the constant rate a forever. Suppose we add a limited carrying capacity term (§11.3), so the first equation in (11.25) becomes

$$\dot{x} = x(a - by - \epsilon x),$$

where $\epsilon > 0$, corresponding to capacity a/ϵ. We will show that the fixed point of this system is a stable focus for small ϵ and a stable node for large ϵ (but still satisfying $\epsilon < ad/c$).

To see this, note that interior equilibrium satisfies $x^* = c/d$ and $y^* = (ad - c\epsilon)/bd$. The Jacobian, evaluated at (x^*, y^*) is

$$J = \begin{bmatrix} -c\epsilon/d & -bc/d \\ y^* & 0 \end{bmatrix}.$$

The eigenvalues of the Jacobian at the equilibrium are

$$\frac{-c\epsilon \pm \sqrt{c}\sqrt{c\epsilon^2 + 4cd\epsilon - 4ad^2}}{2d}.$$

When ϵ is small, the term under the square root sign is negative, so both eigenvalues have negative real parts. The equilibrium in this case is a stable focus. If ϵ is large, but $\epsilon > ad/c$, it is easy to show that both eigenvalues are negative, so the equilibrium is a stable node.

11.12 Take No Prisoners

Two firms share a nonmarketable, nonexcludable resource R that lowers production costs but is subject to overcrowding and depletion. Suppose that when firm 1 has size x and firm 2 has size y, the profits of the two firms are given by

$$\begin{aligned} \pi^x(x, y) &= \gamma_x(R - x - y) - \alpha_x, \\ \pi^y(x, y) &= \gamma_y(R - x - y) - \alpha_y, \end{aligned}$$

where $\gamma_x, g_y > 0$. Suppose also that the firms' growth rates are equal to their profit rates, so

$$\dot{x} = x(\gamma_x(R - x - y) - \alpha_x),$$
$$\dot{y} = y(\gamma_y(R - x - y) - \alpha_y).$$

We assume that $\gamma_x R > \alpha_x$, $\gamma_y R > \alpha_y$, and $\gamma_x/\alpha_x > \gamma_y/\alpha_y$.

a. Show that if $y = 0$ the model has an unstable fixed point at $x = 0$ and an asymptotically stable fixed point at $x = x^* = R - a_x/g_x$.
b. Show that if $x = 0$ the model has an unstable equilibrium at $y = 0$ and an asymptotically stable fixed point at $y = y^* = R - a_y/g_y$.
c. Show that the complete model has three fixed points, (0,0), $(x^*, 0)$, and $(0, y^*)$, of which only the second is asymptotically stable.

We conclude that both firms cannot coexist in equilibrium.

11.13 The Hartman-Grobman Theorem

The linearization theorem 11.3 tells us that we can determine whether a hyperbolic fixed point of a dynamical system is asymptotically stable or unstable by looking at its linearization. This is a fairly weak statement, because we have discovered a lot more about the nature of equilibria than just stability. We have, for instance, distinguished nodes, foci, and saddles, and in the latter case, we have found that there are always stable and unstable manifolds. It turns out that in the hyperbolic case, each of these properties of the linearization of a dynamical system is also possessed by the system itself. This is the famous *Hartman-Grobman theorem.* To state the theorem, however, we need a new definition.

Suppose the dynamical system defined by $\dot{\mathbf{x}} = f(\mathbf{x})$ has a fixed point at $\mathbf{x}^*$, and the dynamical system defined by $\dot{\mathbf{y}} = f(\mathbf{y})$ has a fixed point at $\mathbf{y}^*$. We say that the two systems are *topologically equivalent* at these fixed points if there are balls $B_\epsilon(\mathbf{x}^*)$ and $B_\delta(\mathbf{y}^*)$ around the two fixed points and a continuous one-to-one mapping $\phi : B_\epsilon(\mathbf{x}^*) \to B_\delta(\mathbf{y}^*)$ with a continuous inverse that takes trajectories of the dynamical system lying in $B_\epsilon(\mathbf{x}^*)$ into trajectories of the dynamical system lying in $B_\delta(\mathbf{y}^*)$ and preserves the direction of time.

Intuitively, two dynamical systems are topologically equivalent at $\mathbf{x}^*$ and $\mathbf{y}^*$ if we can perform the following operation. Draw the phase diagram in the neighborhood of $\mathbf{x}^*$ on a rubber sheet, including trajectories and arrows indicating the direction of time. Now stretch the rubber sheet without tearing or folding until it looks just like the phase diagram for the second dynamical system in a neighborhood of $\mathbf{y}^*$. If this is possible, then the systems are topologically equivalent.

THEOREM 11.7 Hartman-Grobman. *If* $\mathbf{x}^*$ *is a hyperbolic fixed point of the dynamical system given by* $\dot{\mathbf{x}} = f(\mathbf{x})$, *then this fixed point is topologically equivalent to the fixed point at the origin of the linearization of the system* $\dot{\mathbf{x}} = A\mathbf{x}$, *where* $A = Df(\mathbf{x}^*)$ *is the Jacobian matrix of the system evaluated at* $\mathbf{x}^*$.

This means that we can determine the qualitative behavior of a dynamical system in a neighborhood of a hyperbolic fixed point by looking at its linearization, which is of course much easier to analyze. Indeed, we have fully characterized such equilibria for one- and two-dimensional systems. Higher-dimensional linear systems are harder to analyze, but they too can be completely characterized and are essentially combinations of one- and two-dimensional systems, placed at angles to each other in higher dimensions.

11.14 Features of Two-Dimensional Dynamical Systems

Two-dimensional dynamical systems have lots of nice properties not shared by higher-dimensional systems. This appears to be due to the famous *Jordan curve theorem*, which says that any continuous, non-self-intersecting, closed curve in the plane divides the plane into two connected pieces—an "inside" and an "outside." Trajectories of a dynamical system are of course continuous and non-self-intersecting, though not generally closed.

Let $\mathbf{x}(t)$ be a trajectory of the dynamical system (11.1). We say a point $\mathbf{y} \in \mathbf{R}^n$ is an ω-limit point of the trajectory if there is a sequence $t_n \to \infty$ such that

$$\lim_{n\to\infty} \mathbf{x}(t_n) = \mathbf{y}.$$

For instance, if $\mathbf{x}^*$ is an asymptotically stable fixed point, then $\mathbf{x}^*$ is the ω-limit of every trajectory starting at a point in the basin of attraction of $\mathbf{x}^*$. In some cases, a trajectory can actually have lots of ω-limit points. For instance, suppose the fixed point $\mathbf{x}^*$ is an unstable spiral, but there is a closed orbit at some distance from $\mathbf{x}^*$. Then a trajectory starting at a point near $\mathbf{x}^*$ can spiral out, getting closer and closer to the closed orbit. Each point on the closed orbit is thus an ω-limit of the trajectory. If a trajectory is bounded (that is, is contained in some ball), then it must have at least one ω-limit point.

THEOREM 11.8 Poincaré-Bendixson. *Suppose* $\mathbf{x}(t)$ *is a bounded trajectory of (11.13), and* Ω *is the set of* ω*-limit points of the trajectory. Then if* Ω *contains no fixed points of (11.13),* Ω *is a periodic orbit of (11.13).*

The following theorem is also often useful.

THEOREM 11.9 *Suppose equation (11.13) has a closed orbit* Γ *and let* U *be the interior region bounded by* Γ*. Then* U *contains a fixed point of (11.13).*