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Answers

Probability Theory: Answers

1.7 Craps

Roller wins immediately with any one of (16,25,34,43,52,61,56,65), which have probability 8/36. Let $p(4)$ be the probability of rolling a 4. Because there are three ways to roll a 4 (13, 22, 31) out of 36 possible rolls, $p(4) = 3/36$. If the Roller first rolls 4, let $q(4)$ be the probability he wins. The ways of rolling a 2, 7, or 12 are (11, 61, 52, 43, 34, 25, 16, 66), so the probability of "crapping out" is 8/36. This gives us the equation

$$q(4) = 3/36 + (1 - 3/36 - 8/36)q(4)$$

because if you do not crap out or roll a 4, the probability of rolling a 4 before crapping out is still $q(4)$. We can solve this for $q(4)$, getting $q(4) = 3/11$. Thus, the probability Roller wins by first rolling a 4 is $p(4)q(4) = 9/396$. We have $p(5) = 4/36$ and $q(5) = 4/36+(1-4/36-8/36)q(5)$, so $q(5) = 4/12$, and the probability of winning by first throwing a 5 is $p(5)q(5) = 16/432$. Similarly, $p(6) = 5/36$ and $q(6) = 5/13$, so the probability of winning by first throwing a 6 is 25/468. Also, $p(8) = p(6) = 5/36$, so the probability of winning by first throwing an 8 is also 25/468. Again, $p(9) = p(5)$, so the probability of winning by first throwing a 9 is 16/432. Finally $p(10) = p(4)$, so the probability of winning by first throwing a 10 is 9/396. Thus, the probability of winning is

$$8/36+9/396+16/432+25/468+24/468+16/432+9/396 = 6895/15444,$$

or about 44.645%. So, you can see why the casino likes Craps. But, why does Roller like Craps?

1.8 A Marksman Contest

If Alice plays Bonnie twice, she wins the contest with probability $pq + (1-p)qp = pq(2-p)$, but if she plays Carole twice, she wins $qp+(1-q)pq = pq(2-q)$, which is larger.

1.9 Sampling

A die has six possible outcomes. Throwing six dice is like sampling one die six times with replacement. Thus, there are $6^6 = 46656$ ordered configurations of the 6 dice. There are 6 outcomes in which all the faces are the same. Thus, the probability is $6/46656 = 0.0001286$. A more straightforward solution is to note that the second through sixth die must match the first, which happens with probability $(1/6)^5$.

1.10 Aces Up

There are 52 ways to choose the first card, and 51 ways to choose the second card, so there are 52×51 different ways to choose two cards from the deck. There are 4 ways to choose the first ace, and 3 ways to choose the second, so there are 4×3 ways to choose a pair of aces. Thus, the probability of choosing a pair of aces is $12/(52 \times 51) = 1/221 \approx 0.0045248 \approx 0.45248\%$.

1.11 Permutations

Let's first solve the problem for a particular n, say $n = 3$. We can write the $n! = 6$ permutations as follows:

$$\begin{array}{cccccc} 1 & 1 & 2 & 2 & 3 & 3 \\ 2 & 3 & 1 & 3 & 1 & 2 \\ 3 & 2 & 3 & 1 & 2 & 1 \end{array}$$

Each row has exactly $2 = (n-1)!$ matches and there are $3 = n$ rows, so the total number of matches is $6 = n \times (n-1)! = n!$. Thus the average number of matches is $6/6 = n!/n! = 1$. You can generalize this to show that the average number of matches for any n is 1.

1.13 Mechanical Defects

This is sampling 2 times without replacement from a set of 7 objects. There are $7!/(7-2)! = 7 \times 6 = 42$ such samples. How many of these are two nondefective machines? How many samples of two are there from a population of 5 (the number of nondefectives)? The answer is $5!/(5-2)! = 5 \times 4 = 20$. Thus, the probability is $20/42 = 0.4762$.

1.14 Mass Defection

The number of ways of selecting 10 items from a batch of 100 items equals the number of combinations of 100 things taken 10 at a time, which is 100!/10!90!. If the batch is accepted, all of the 10 items must have been chosen from the 90 nondefective items. The number of such combinations of ten items is 90!/10!80!. Thus, the probability of accepting the batch is

$$\begin{aligned}\frac{(90!/10!80!)}{(100!/10!90!)} &= \frac{90!90!}{80!100!}\\ &= \frac{90\times 89\times\ldots\times 81}{100\times 99\times\ldots 91},\end{aligned}$$

which is approximately 33.04%.

1.15 House Rules

Here is an equivalent game: you ante \$1,000 and choose a number. The house rolls the three dice, and pays you \$2,000 for one match, \$3,000 for two matches, and \$4,000 for three matches. The probability of one match is

$$\binom{3}{1}\frac{1}{6}\times\frac{5}{6}\times\frac{5}{6}=\frac{75}{216},$$

the probability of two matches is

$$\binom{3}{2}\frac{1}{6}\times\frac{1}{6}\times\frac{5}{6}=\frac{15}{216},$$

and the probability of three matches is 1/216. The expected payoff is thus

$$2000\frac{75}{216}+3000\frac{15}{216}+4000\frac{1}{216}=\frac{19900}{216}=921.3.$$

Thus, you can expect to lose \$78.70 every time you play.

1.17 A Guessing Game

Suppose the first guess is k. This is correct with probability $1/n$, high with probability $(k-1)/n$, and low with probability $(n-k)/n$. Thus, the expected number of guesses given that the first guess is k is given by

$$f(n|k) = \frac{1}{n} + \frac{(k-1)[1+f(k-1)]}{n} + \frac{(n-k)[1+f(n-k)]}{n},$$

where $f(0) = 0$. But we also know that

$$f(n) = f(n|1)/n + ... + f(n|n)/n.$$

Thus, we have

$$\begin{aligned} f(n) &= \frac{1}{n} + \sum_{k=1}^{n}(k-1)[1+f(k-1)]/n^2 + \sum_{k=1}^{n}(n-k)[1+f(n-k)]/n^2 \\ &= \frac{1}{n} + \sum_{k=1}^{n}[n-1+(k-1)f(k-1)+(n-k)f(n-k)]/n^2 \\ &= 1 + \frac{2}{n^2}\sum_{k=1}^{n-1} kf(k). \end{aligned}$$

Let us solve this recursive equation. Note that

$$\begin{aligned} f(n) &= 1 + \frac{2}{n^2}\left[f(1) + 2f(2) + ... + (n-1)f(n-1)\right] \\ &= 1 + \frac{2(n-1)}{n^2}f(n-1) \\ &\quad + \frac{(n-1)^2}{n^2}\frac{2}{(n-1)^2} \\ &\quad \times[f(1) + 2f(2) + ... + (n-2)f(n-2)] \\ &= 1 + \frac{2(n-1)}{n^2}f(n-1) + \frac{(n-1)^2}{n^2}[f(n-1)-1]. \end{aligned}$$

Collecting terms and rearranging a bit, we have

$$\frac{nf(n)-3}{n+1} = \frac{(n-1)f(n-1)-3}{n} + \frac{2}{n}.$$

If we write $g(n) = [nf(n)-3]/(n+1)$, the last equation becomes

$$g(n) = g(n-1) + \frac{2}{n},$$

with $g(1) = [f(1) - 3]/2 = -1$. Thus,

$$g(n) = -3 + 2\sum_{k=1}^{n} k^{-1}.$$

Finally,

$$f(n) = \frac{n+1}{n}\left[-3 + 2\sum_{k=1}^{n} k^{-1}\right] + \frac{3}{n}.$$

We can approximate $f(n)$ for large n by noting that

$$\sum_{k=1}^{n} k^{-1} = \frac{3}{2} \approx \frac{3}{2} + \int_{3}^{n} \frac{dk}{k} = \frac{3}{2} + \ln\left(\frac{n}{3}\right).$$

Thus,

$$f(n) \approx \frac{n+1}{n} \ln\left(\frac{n}{3}\right) + \frac{3}{n} \approx \ln\left(\frac{n}{3}\right) \approx \ln(n).$$

for large n.

1.18 North Island, South Island

Let P_s be the probability of finding the treasure if Bob is on South Island. Then we have

$$P_n = q_n + r_n P_s + (1 - q_n - r_n)P_n$$

and

$$P_s = e_s P_s + (1 - e_s - r_s)P_n.$$

Now, solve these two equations for P_n.

1.21 Extrasensory Perception

Suppose Alice's first draw, a_1, is less than the other player's draw, b. Then the probability Alice's next draw, a_2, is higher than a_1 is given by:

$$\mathrm{P}[a_2 > a_1 | b > a_1] = \frac{\mathrm{P}[a_2 > a_1 \wedge b > a_1]}{\mathrm{P}[b > a_1]}.$$

The numerator in this expression is equal to the probability that a_1 is the lowest of three draws, which is 1/3, and the denominator is equal to the probability that a_1 is the lowest of two draws, which is 1/2. Thus, Alice beats herself on the second draw with probability 2/3, and the overall probability she wins is $(1/2) + (1/2)(2/3) = 5/6$.

1.22 Les Cinq Tiroirs

We depict the whole event space as a rectangle with six pieces. Piece A, which consists of 20% of the space, represents the event "the object is not in any drawer."

A	D1	16%
	D2	16%
	D3	16%
	D4	16%
20%	D5	16%

The other five events, D1, D2, D3, D4, and D5, represent the event where the object is in one of the drawers. Because these are equally likely, each such event represents (1-0.2)/5 = 16% of the space.

The probability of D1, which we may write P$[D1]$ is, of course 16%. The probability of D2 given not D1 is P$[D2|D1^c]$. We can evaluate this by

$$\mathrm{P}[D2|D1^c] = \frac{\mathrm{P}[D2 \wedge D1^c]}{\mathrm{P}[D1^c]} = \frac{\mathrm{P}[D2]}{1-0.16} = 0.16/0.84 \approx 19\%.$$

The probability of D3 given not D1 or D2 is P$[D3|D1^c \wedge D2^c]$. We can evaluate this by

$$\mathrm{P}[D3|D1^c \wedge D2^c] = \frac{\mathrm{P}[D3 \wedge D1^c \wedge D2^c]}{\mathrm{P}[D1^c \wedge D2^c]} = \frac{\mathrm{P}[D3]}{1-0.16-0.16}$$

$$= 0.16/0.68 \approx 23.5\%.$$

You can check that the probability of finding the object in the fourth drawer, given that it was not in any previous drawer, is $0.16/0.52 = 30.77\%$, and the probability that it is in the fifth drawer given that it is neither of the first four is $0.16/0.36 = 44.44\%$. So the probability of finding the object rises from drawer 1 to drawer 5.

What about the probability of not finding the object? Let N be the event "the object is in none of the drawers." the P$[N] = 0.2$. What is P$[N|D1^c]$, the probability it is none of the drawers if it is not in the first drawer. Well, by definition of conditional probability,

$$\mathrm{P}[N|D1^c] = \frac{\mathrm{P}[N \wedge D1^c]}{\mathrm{P}[D1^c]} = \frac{\mathrm{P}[N]}{\mathrm{P}[D1^c]} = 0.2/0.84 = 23.81\%.$$

The probability the object is in none of the drawers if it is found not to be in either of the first two is, similarly (do the reasoning!) $0.2/0.68 = 29.41\%$. It is easy now to do the rest of the problem (do it!).

1.23 Drug Testing

We have $\mathrm{P}[A] = 1/20$ and $\mathrm{P}[\mathrm{Pos}|A] = \mathrm{P}[\mathrm{Neg}|A^c] = 19/20$. Thus,

$$\begin{aligned}\mathrm{P}[A|\mathrm{Pos}] &= \frac{\mathrm{P}[\mathrm{Pos}|A]\,\mathrm{P}[A]}{\mathrm{P}[\mathrm{Pos}|A]\,\mathrm{P}[A] + \mathrm{P}[\mathrm{Pos}|A^c]\,\mathrm{P}[A^c]} \\ &= \frac{\mathrm{P}[\mathrm{Pos}|A]\,\mathrm{P}[A]}{\mathrm{P}[\mathrm{Pos}|A]\,\mathrm{P}[A] + (1 - \mathrm{P}[\mathrm{Neg}|A^c])\mathrm{P}[A^c]} \\ &= \frac{(19/20)(1/20)}{(19/20)(1/20) + (19/20)(1/20)} = 1/2.\end{aligned}$$

We can answer the problem without using Bayes' rule just by counting. Suppose we test 10,000 people (the number does not matter). Then $10,000 \times 0.05 = 500$ use drugs (on average), of whom $500 \times 0.95 = 475$ test positive (on average). But 9,500 do not use drugs (again, on average), and $9,500 \times (1 - 0.95) = 475$ also test positive (on average). Thus of the 950 ($= 475 + 475$) who test positive, exactly 50% use drugs (on average).

1.25 Urns

For any $k = 0, \ldots, n$, let p_k^s be the probability that you are drawing from the kth urn, given then you have drawn s red balls from the urn. Let R^s be the event "drew s red balls from the urn," and let U_k be the event "you are drawing from urn k." Then, we have

$$p_k^s = \frac{P[R^s|U_k]P[U_k]}{\sum_{i=0}^n P[R^s|U_i]P[U_i]} = \frac{P[R^s|U_k]}{\sum_{i=0}^n P[R^s|U_i]} = \frac{k^s}{\sum_{i=0}^n i^s}.$$

Let R be the event "the next ball drawn is red." Then, the probability the next ball will be red, given that we have already drawn s red balls, which we can write $P[R|R^s]$, is given by

$$\begin{aligned} P[R|R^s] &= \sum_{i=0}^{n} P[R|U_i]p_i^s = \sum_{i=0}^{n} P[R|U_i]\frac{i^s}{\sum_{j=0}^{n} j^s} \\ &= \sum_{i=0}^{n} \frac{i}{n+1}\frac{i^s}{\sum_{j=0}^{n} j^s} = \frac{\sum_{j=0}^{n} j^{s+1}}{(n+1)\sum_{j=0}^{n} j^s}. \end{aligned}$$

These expressions have closed form evaluations, but we can approximate them more easily by integrals. Thus,

$$\sum_{i=0}^{n} i^s \approx \int_0^n x^s dx = n^{s+1}/(s+1).$$

Thus,

$$P[R|R^s] \approx \frac{n^{s+2}}{s+2}\frac{1}{n+1}\frac{s+1}{n^{s+1}} = \frac{n}{n+1}\frac{s+1}{s+2}.$$

1.26 The Monty Hall Game

Let p be the event that the contestant chooses the winning door, say door A, so $\mathrm{P}[p] = 1/3$. Let q be the event that Monty Hall chooses a door, say door B, from among the other two doors, and door B has no prize behind it. From Bayes' rule, we have

$$\mathrm{P}[p|q] = \frac{\mathrm{P}[q|p]\mathrm{P}[p]}{\mathrm{P}[q]}.$$

But $\mathrm{P}[q|p] = 1$, because Monty Hall cannot choose door A, so if p holds, than q must also hold. Thus we have

$$\mathrm{P}[p|q] = \frac{1}{3\mathrm{P}[q]}.$$

If Monty Hall chose a door that he *knew* has no prize behind it, then $\mathrm{P}[q] = 1$, so $\mathrm{P}[p|q] = 1/3$. The probability that the prize is behind door C is then $1 - 1/3 = 2/3$, so the contestant doubles the probability of winning the prize by shifting from door A to door C. However, if Monty Hall chooses *randomly* between doors B and C, then $\mathrm{P}[q] = 2/3$, so $\mathrm{P}[p|q] = 1/2$. The probability that the prize is behind door C is then $1 - 1/2 = 1/2$, so the contestant cannot gain from shifting.

It is instructive to generalize this to n doors. The contestant chooses a door, say A, and the event q is now that Monty Hall opens all the other doors but one, and none has a prize behind it. Does the contestant gain from switching?

We now have $\mathrm{P}[p] = 1/n$ and $\mathrm{P}[q|p] = 1$. Thus $\mathrm{P}[p|q] = 1/n\mathrm{P}[q]$. If Monty Hall always chooses a door with no prize behind it, then $\mathrm{P}[q] = 1$, so $\mathrm{P}[p|q] = 1/n$, and the probability that the prize is behind the remaining door is then $1 - 1/n = (n-1)/n$. Thus, for $n \geq 3$, the contestant gains by switching. However, if Monty Hall chose randomly, then $\mathrm{P}[q] = (n-2)/n$. This is because the probability that the prize is behind one of the two doors he did not choose is just $2/n$. In this case, then, $P[p|q] = 1/(n-2)$, so the probability the prize is behind the other unopened door is $(n-1)/(n-2) > 1/(n-2)$, so the contestant gains (a lot!) from shifting.

1.27 The Logic of Murder and Abuse

First, from Bayes' rule,

$$\mathrm{P}[C|A] = \frac{\mathrm{P}[A|C]\,\mathrm{P}[C]}{\mathrm{P}[A]}.$$

This is the probability that a man murders his wife if he has abused her. But from (d) above, $\mathrm{P}[A|C] = 9/10$; from (c) $\mathrm{P}[C] = 1/4000$; from (a), $\mathrm{P}[A] = 1/20$; so we find $\mathrm{P}[C|A] = 4.50\%$.

"I object!" says the chief prosecutor. "The defense ignores the fact that Nicole was *murdered*. What we *really* must know is $\mathrm{P}[C|AB]$, the probability a *murdered* woman who was abused by her husband was murdered by him." "But," splutters the astounded judge, "how could you calculate such a complex probability?" A computer projector is brought into the court, and the chief prosecutor reveals the following calculation, the astute jurors taking mental notes. "We have," says the prosecutor,

$$\mathrm{P}[C|AB] = \frac{\mathrm{P}[ABC]}{\mathrm{P}[AB]} = \frac{\mathrm{P}[AC]}{\mathrm{P}[ABC] + \mathrm{P}[ABC^c]} =$$

$$\frac{\mathrm{P}[A|C]\,\mathrm{P}[C]}{\mathrm{P}[AC]+\mathrm{P}[A|BC^c]\,\mathrm{P}[BC^c]}=\frac{\mathrm{P}[A|C]\,\mathrm{P}[C]}{\mathrm{P}[A|C]\,\mathrm{P}[C]+\mathrm{P}[A](\mathrm{P}[B]-\mathrm{P}[C])},$$

where $\mathrm{P}[A|BC^c]=\mathrm{P}[A]$ by (e). From (b), $\mathrm{P}[B]=1/200$, so $\mathrm{P}[C|B]=\mathrm{P}[C]/\mathrm{P}[B]=1/2$, so we have $\mathrm{P}[C|AB]=18/19=94.74\%$.

1.29 The Greens and the Blacks

Let A be the event "A bridge hand contains at least two aces." Let B be the event "A bridge hand contains at least one ace." Let C be the event "A bridge hand contains the ace of spades."

Then $\mathrm{P}[A|B]$ is the probability that a hand contains two aces if it contains one ace and hence is the first probability sought. Also $\mathrm{P}[A|C]$ is the probability a hand contains two aces if it contains the ace of spades, which is the second probability sought. By Bayes' rule,

$$\mathrm{P}[A|B]=\frac{\mathrm{P}[AB]}{\mathrm{P}[B]}=\frac{\mathrm{P}[A]}{\mathrm{P}[B]}\quad\text{and}\quad\mathrm{P}[A|C]=\frac{\mathrm{P}[AC]}{\mathrm{P}[C]}.$$

Clearly, $\mathrm{P}[C]=0.25$, because all four hands are equally likely to get the ace of spades.

To calculate $\mathrm{P}[B]$, note that the total number of hands with no aces is the number of ways to take 13 objects from 48 (the 52 cards minus the four aces), which is $\binom{48}{13}$.

The probability of a hand having at least one ace is then

$$\mathrm{P}[B]=\frac{\binom{52}{13}-\binom{48}{13}}{\binom{52}{13}}=1-\frac{39\times 38\times 37\times 36}{52\times 51\times 50\times 49}=0.6962.$$

The probability of at least two aces is the probability of at least one ace minus the probability of exactly one ace. We know the former, so let's calculate the latter.

The number of hands with exactly one ace is four times $\binom{48}{12}$, because you can choose the ace in one of four ways, and then choose any combination of 12 cards from the 48 non-aces. But

$$\frac{4\times\binom{48}{12}}{\binom{52}{13}}=\frac{39\times 38\times 37}{51\times 50\times 49}\approx 0.4388,$$

which is the probability of having exactly one ace. The probability of at least two aces is thus

$$\mathrm{P}[A] = .6962 - .4388 = .2574$$

(to four decimal places).

Now $\mathrm{P}[AC]$ is the probability of two aces including the ace of spades. The number of ways to get the ace of spades plus one other ace is calculated as follows: take the ace of spades out of the deck, and form hands of twelve cards. The number of ways of getting no aces from the remaining cards is $\binom{48}{12}$, so the number of hands with one other ace is $\binom{51}{12} - \binom{48}{12}$. The probability of two aces including the ace of spades is thus

$$\frac{\binom{51}{12} - \binom{48}{12}}{\binom{52}{13}} = .1402.$$

Thus, $\mathrm{P}[AC] = .1402$. We now have

$$\mathrm{P}[A|C] = \frac{\mathrm{P}[AC]}{\mathrm{P}[C]} = \frac{.1402}{.25} = .5608 > \frac{\mathrm{P}[AB]}{\mathrm{P}[B]} = \frac{.2574}{.6962} = .3697.$$

1.30 The Brain and Kidney Problem

Let A be the event "the jar contains two brains," and let B be the event "the mad scientist pulls out a brain." Then $\mathrm{P}[A] = \mathrm{P}[A^c] = 1/2$, $\mathrm{P}[B|A] = 1$, and $\mathrm{P}[B|A^c] = 1/2$. Then from Bayes' rule, the probability that the remaining blob is a brain is $\mathrm{P}[A|B]$, which is given by

$$\mathrm{P}[A|B] = \frac{\mathrm{P}[B|A]\mathrm{P}[A]}{\mathrm{P}[B|A]\mathrm{P}[A] + \mathrm{P}[B|A^c]\mathrm{P}[A^c]} = \frac{1/2}{1/2 + (1/2)(1/2)} = 2/3.$$

1.31 The Value of Eyewitness Testimony

Let G be the event "Cab that hit Alice was green," let B be the event "cab that hit Alice was blue," let WB be the event "witness records seeing blue cab," and finally, let WG be the event "witness records seeing green cab." We have $\mathrm{P}[G] = 85/100 = 17/20$, $\mathrm{P}[B] = 15/100 = 3/20$, $\mathrm{P}[WG|G] = \mathrm{P}[WB|B] = 4/5$, $\mathrm{P}[WB|G] = \mathrm{P}[WG|B] = 1/5$. Then Bayes' rule yields

$$\mathrm{P}[B|WB] = \frac{\mathrm{P}[WB|B]\mathrm{P}[B]}{\mathrm{P}[WB|B]\mathrm{P}[B] + \mathrm{P}[WB|G]\mathrm{P}[G]},$$

which evaluates to 12/29.

1.32 When Weakness Is Strength

Suppose a player is picked randomly to shoot in each round. It remains true in this case that Alice and Bob will shoot at each other until only one of them remains. However, clearly Carole now prefers to have a one-on-one against Bob rather than against Alice, so Carole will shoot at Alice if given the chance. Now

$$\pi_a(ab) = \frac{1}{2} + \frac{1}{2} \times \frac{1}{5}\pi_a(ab),$$

so $\pi_a(ab) = 5/9$ and $\pi_b(ab) = 4/9$. Similar reasoning gives $\pi_a(ac) = 2/3$ and $\pi_c(ac) = 1/3$. Finally,

$$\pi_b(bc) = \frac{1}{2}\left(\frac{4}{5} + \frac{1}{5}\pi_b(bc)\right) + \frac{1}{2} \times \frac{1}{2}\pi_b(bc),$$

from which we conclude $\pi_b(bc) = 8/13$ and $\pi_c(bc) = 5/13$. Now clearly $\pi_a[a] = \pi_a(ac) = 2/3$, $\pi_b[a] = 0$, and $\pi_c[a] = 1/3$. Similarly, it is easy to check that

$$\begin{aligned}
\pi_b[b] &= (4/5)\pi_b(bc) + (1/5)\pi_b \\
\pi_a[b] &= (1/5)\pi_a \\
\pi_c[b] &= (4/5)pi_c(bc) + (1/5)\pi_c \\
\pi_c[c] &= (1/2)\pi_c(bc) + (1/2)\pi_c \\
\pi_b[c] &= (1/2)\pi_b(bc) + (1/2)\pi_b \\
\pi_a[c] &= (1/2)\pi_a.
\end{aligned}$$

Moving to the final calculations, we have

$$\pi_b = \frac{1}{3}\left[0 + \frac{4}{5}\pi_b(bc) + \frac{1}{5}\pi_b + \frac{1}{2}\pi_b(bc) + \frac{1}{2}\pi_b\right].$$

We can solve this for π_b, getting $\pi_b = 24/69$. The similar equation for marksman Alice is

$$\pi_a = \frac{1}{3}\left[\frac{2}{3} + \frac{1}{5}\pi_a + \frac{1}{2}\pi_a\right],$$

which gives $\pi_a = 20/69$. Finally,

$$\pi_c = \frac{1}{3}\left[\frac{1}{3} + \frac{4}{5}\pi_c(bc) + \frac{1}{5}\pi_c + \frac{1}{2}\pi_c(bc) + \frac{1}{2}\pi_c\right],$$

which gives $\pi_c = 25/69$. Clearly, $\pi_c > \pi_b > \pi_a$, so the meek inherit the earth.

1.33 From Uniform to Exponential

Let p_k be the probability that $n = k$. Then, Alice wins \$$k$ with probability p_k, so her average winnings are

$$\begin{aligned} W &= 2p_2 + 3p_3 + 4p_4 + \ldots \\ &= 2 + p_3 + 2p_4 + \ldots \\ &= 2 + (p_3 + p_4 + \ldots) + (p_4 + p_5 + \ldots) + \ldots \\ &= 2 + \mathrm{P}[n > 2] + \mathrm{P}[n > 3] + \ldots \\ &= 2 + 1/2! + 1/3! + \ldots = e, \end{aligned}$$

where $e \approx 2.71$ is the base of the natural logarithms.

1.34 Laplace's Law of Succession

Suppose there are n balls in the urn, and assume the number of white balls is uniformly distributed between 0 and n. Let A_k be the event "there are k white balls," and let B_{rm} be the event "of m balls chosen with replacement, r are white." Then $\mathrm{P}[A_k] = 1/(n+1)$, and by Bayes' rule we have

$$\mathrm{P}[A_k|B_{rm}] = \frac{\mathrm{P}[B_{rm}|A_k]\mathrm{P}[A_k]}{\mathrm{P}[B_{rm}]}.$$

Now it is easy to check that

$$\mathrm{P}[B_{rm}|A_k] = \binom{m}{r}\left(\frac{k}{n}\right)^r\left(1-\frac{k}{n}\right)^{m-r}$$

and

$$\mathrm{P}[B_{rm}] = \sum_{k=0}^{n} \mathrm{P}[A_k]\mathrm{P}[B_{rm}|A_k]. \tag{A1}$$

The probability of choosing a white ball on the next draw is then

$$\begin{aligned} \sum_{k=0}^{n}\left(\frac{k}{n}\right)\mathrm{P}[A_k|B_{rm}] &= \sum_{k=0}^{n}\frac{k\mathrm{P}[B_{rm}|A_k]}{n(n+1)\mathrm{P}[B_{rm}]} \\ &= \frac{1}{(n+1)\mathrm{P}[B_{rm}]}\binom{m}{r}\sum_{k=0}^{n}\left(\frac{k}{n}\right)^{r+1}\left(1-\frac{k}{n}\right)^{m-r}. \end{aligned}$$

To approximate this expression, note that if n is large, equation (A1) is a Riemann sum representing the integral

$$\mathrm{P}[B_{rm}] \approx \frac{1}{n+1}\binom{m}{r}\int_0^1 x^r(1-x)^{m-r} = \frac{1}{(n+1)(m+1)}, \qquad \text{(A2)}$$

where the integral is evaluated by integration by parts r times. Replacing m by $m+1$ and r by $r+1$ in the preceding expression, we see that equation (AA2) is approximately

$$\frac{1}{(n+1)\mathrm{P}[B_{rm}]}\frac{m!}{r!(m-r)!}\frac{(r+1)!(m-r)!}{(m+2)(m+1)!} = \frac{r+1}{m+2}.$$

Eliminating Dominated Strategies: Answers

4.3 Exercises in Eliminating Dominated Strategies

(c) $N_2 < J_2$, $C_1 < N_1$, $J_2 < C_2$, $N_1 < J_1$.
(d) $C > D$, $e > a$, $B > E$, $c > b$, $B > A$, $c > d$, $B > C$,$c > e$.

4.6 Second-Price Auction

Suppose first you win, and let v_s be the second-highest bid. If you had bid more than v_i, you still would have won, and your gain would still be the same, namely $v_i - v_s \geq 0$. If you had bid lower than v_i, there are three subcases: you could have bid more than, equal to, or less than v_s. If you had bid more than v_s, you would have had the same payoff, $v_i - v_s$. If you had bid equal to v_s, you could have lost the auction in the playoff among the equally high bidders, and if you had bid less than v_s, you certainly would have lost the auction. Hence, nothing beats bidding v_i in case you win.

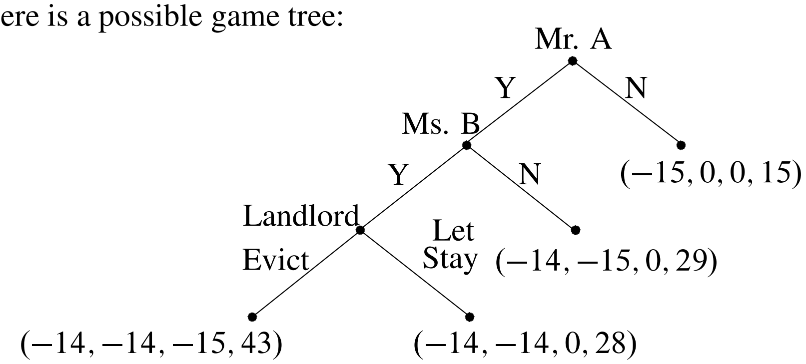
But suppose you bid v_i and lost. Let v_h be the highest bid and v_s be the second-highest bid. Because you lost, your payoff is zero, so if you had bid less than v_i, you would still have lost, so you could not improve your payoff this way. If had you bid more than v_i, it would not matter unless you had bid enough to win the auction, in which case your gain would have been $v_s - v_i$. Because $v_i \neq v_h$, we must have $v_i \leq v_s$, as v_s is the second-highest offer. Thus, you could not have made a positive gain by bidding higher than v_i.

Hence, bidding v_i is a best response to any set of bids by the other players.

a. Because "truth telling" is a dominant strategy, it remains a best response no matter what the other players do.
b. Yes, it could matter. For instance, suppose you are bidder 1 and all other bidders $i = 2, \ldots, n$ follow the strategy of bidding zero first, and bidding \$1 more than the highest bid, provided the highest bid is less that v_i. Then, if you bid an amount greater than the largest v_i for the other players, you win and pay zero. If you bid your value v_1, by contrast, and some $v_i > v_1 + 1$, you will not win the auction.
c. If every player uses truth telling except you, you can bid a very small amount lower that the highest value v_i, ensuring that the winner of the lottery has very small payoff.

4.8 The Eviction Notice

Here is a possible game tree:



4.9 Hagar's Battles

Each side should deploy its troops to the most valuable battlefields. To see this, suppose player 1 does not. Let x_j be the highest value battlefield unoccupied by player 1, and let x_i be the lowest value battlefield occupied by player 1. What does player 1 gain by switching a soldier from x_i to x_j? If both are occupied by player 2, there is no change. If neither is occupied by player 2, player 1 gains $a_j - a_i > 0$. If player 2 occupies x_j but not x_i, player 1 loses a_i by switching, and player 2 loses a_j, so player 1 gains $a_j - a_i > 0$. Similarly if player 2 occupies x_i but not x_j.

Another explanation: Suppose you occupy a_i but not a_j, where $a_j > a_i$. The figure below shows that the gain from switching from a_i to a_j is positive in all contingencies.

Enemy Occupies

	a_i not a_j	a_j not a_i	a_i and a_j	neither
loss	lose i	lose i	lose i	lose i
gain	gain j	gain j	gain j	gain j
net gain	$j - i$	$j - i$	$j - i$	$j - i$

4.10 Military Strategy

First we can eliminate all country I strategies that do not arrive at A. This leaves six strategies, which we can label fcb, feb, fed, hed, heb, and hgd.

We can also eliminate all country A strategies that stay at A at any time, or that hit h or f. This leaves the six strategies bcb,beb,bed,ded,deb,dgd. The payoff matrix is:

	bcb	beb	bed	ded	deb	dgd
fcb	−1	−1	1	1	−1	1
feb	−1	−1	−1	−1	−1	1
fed	1	−1	−1	−1	−1	−1
hed	1	−1	−1	−1	−1	−1
heb	−1	−1	−1	−1	−1	1
hgd	1	1	−1	−1	1	−1

Now feb is weakly dominated by fcb, as is heb. Moreover, we see that fed and hed are weakly dominated by hgd. Thus there are two remaining strategies for country I, "south" (hgd) and "north" (fcb).

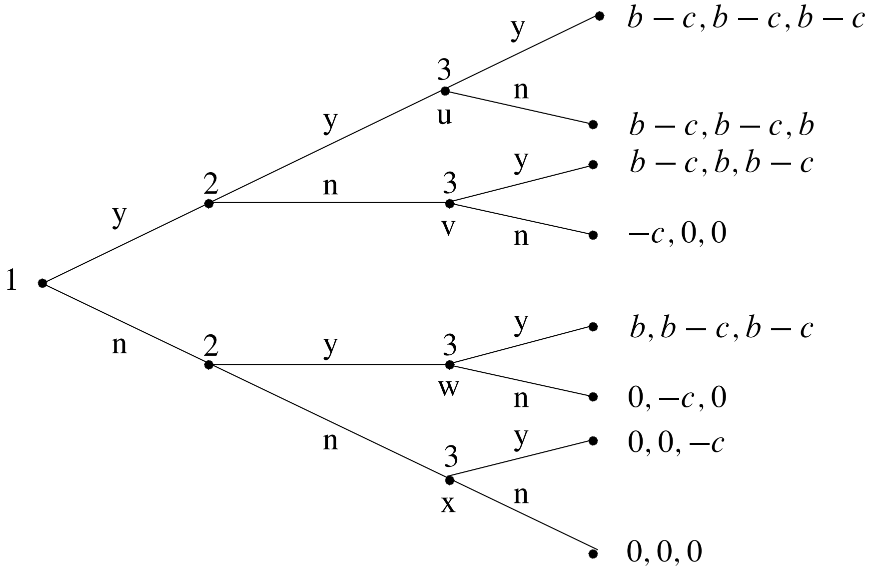
Also bcb is dominated by beb and dgd is dominated by ded, so we may drop them. Moreover, beb and deb are the same "patrol north," whereas bed and ded are the same "patrol south." This gives us the reduced game:

	patrol north	patrol south
attack north	−1,1	1,−1
attack south	1,−1	−1,1

So this complicated game is just the heads-tails game, which we will finish solving when we do mixed-strategy equilibria!

4.12 Strategic Voting

We can solve this by pruning the game tree. We find that player 1 chooses no, and players 2 and 3 choose yes, with payoff $(b, b-c, b-c)$. It is best to go first. The game tree is the following:



Note that this does not give a full specification of the strategies, because player 2 has 4 strategies and player 3 has 16 strategies. The preceding description says only what players 2 and 3 do "along the game path," that is, as the game is actually played.

To describe the Nash equilibrium in full, let us write player 3's strategies as "uvwx," where u, v, w, and x are each either y (yes) or n (no) and indicate the choice at the corresponding node in the game tree, starting from the top. Then the third player's choice is nyyn. Similarly, player 2's choice is ny, and the first player's is, of course, n.

If player 3 chooses nnnn, player 2 chooses yn, and player 1 chooses y, we have another Nash equilibrium (check it out!), in which player 3 now gets b and the other two get $b-c$. The equilibrium is strange because it means that player 3 should make suboptimal choices at nodes v and w; he says he will choose "no" but in fact he will choose "yes" at these nodes, because this gives him a higher payoff. The strategy nnnn is called an *incredible threat*, because it involves player 3 threatening to do something that he in fact will not do when it comes time to do it. But if the others believe him, he will never have to carry out his threat! We say such a Nash equilibrium violates subgame perfection.

Pure-Strategy Nash Equilibria: Answers

5.4 The Tobacco Market

a. Let's not use numbers until we need to. We can write $p = a - bq$, where $q = q_1 + q_2 + q_3$, and q_i is the amount sent to market by farmer i. Farmer 1 maximizes $pq_1 = (a - bq)q_1$. If there is an interior solution, the first-order condition on q_1 must satisfy

$$a - b(q_2 + q_3) - 2bq_1 = 0.$$

If all farmers ship the same amount of tobacco, then $q_2 = q_3 = q_1$, so this equation becomes $4bq_1 = a$, which gives $q_1 = q_2 = q_3 = a/4b$, and $q = 3a/4b$, so $p = a/4$. The revenue of each farmer is $pq = a^2/16b$. In our case $a = 10$ and $b = 1/100000$, so the price is \$2.50 per pound, and each farmer ships 250,000 pounds and discards the rest. The price support does not matter, because $p >$ \$0.25. Each farmer has profit \$625,000.

If the second and third farmers send their whole crop to market, then $q_2 + q_3 = 1,200,000$. In this case even if farmer 1 shipped nothing, the market price would be $10 - 1,200,000/100,000 = -2 < 0.25$, so the price support would kick in. Farmer 1 should then also ship all his tobacco at \$0.25 per pound, and each farmer has profit \$150,000.

b. You can check that there are no other Nash equilibria. If one farmer sends all his crop to market, the other two would each send 400,000/3 pounds to market. But then the first farmer would gain by sending less to market.

5.5 The Klingons and the Snarks

Suppose the Klingons choose a common rate r of consumption. Then each eats 500 snarks, and each has payoff

$$u = 2000 + 50r - r^2.$$

Setting the derivative u' to zero, we get $r = 25$, so each has utility $u = 2000 + 50(25) - 25^2 = 2625$.

Now suppose they choose their rates separately. Then

$$u_1 = \frac{4000r_1}{r_1 + r_2} + 50r_1 - r_1^2.$$

Setting the derivative of this to zero, we get the first-order condition

$$\frac{\partial u_1}{\partial r_1} = \frac{4000r_2}{(r_1 + r_2)^2} + 50 - 2r_1 = 0,$$

and a symmetrical condition holds for the second Klingon:

$$\frac{\partial u_2}{\partial r_2} = \frac{4000r_1}{(r_1 + r_2)^2} + 50 - 2r_2 = 0.$$

These two imply

$$\frac{r_2}{r_1} = \frac{r_1 - 25}{r_2 - 25},$$

which has solutions $r_1 = r_2$ and $r_1 + r_2 = 25$. The latter, however, cannot satisfy the first-order conditions. Setting $r_1 = r_2$, we get

$$\frac{4000}{4r_1} + 50 - 2r_1 = 0,$$

or $1000/r_1 + 50 - 2r_1 = 0$. This is a quadratic that is easy to solve. Multiply by r_1, getting $2r_1^2 - 50r_1 - 1000 = 0$, with solution $r = (50 + \sqrt{(}2500 + 8000))/4 = 38.12$. So the Klingons eat about 50% faster than they would if they cooperated! Their utility is now $u = 2000 + 50r_1 - r_1^2 = 2452.87$, lower than if they cooperated.

5.6 Chess: The Trivial Pastime

We will have to prove something more general. Let's call a game *Chessian* if it is a finite game of perfect information in which players take turns, and the outcome is either (win,lose), (lose,win), or (draw,draw), where win is preferred to draw, and draw is preferred to lose. Let us call a game *certain* if it has a solution in pure strategies. If a Chessian game is certain, then clearly either one player has a winning strategy or both players can force a draw. Suppose there were a Chessian game that is not certain. Then there must be a *smallest* Chessian game that is not certain (that is, one with fewest nodes). Suppose this has k nodes. Clearly, $k > 1$, because it is obvious that a Chessian game with one node is certain. Take any node all of whose child nodes are terminal nodes (why must this exist?). Call this node A. Suppose Red (player 1) chooses at A (the argument is similar if Black chooses). If one of the terminal nodes from A is (win,lose), label A (lose,win); if all of the terminal nodes from A are (lose,win), label A (win,lose); otherwise label A (draw,draw). Now erase the branches from A, along with their terminal nodes. Now we have a new, smaller, Chessian game, which is certain, by our induction assumption. It is easy to see that if Red has a winning strategy in the smaller game, it can be extended to a winning strategy in the larger game. Similarly, if Black has a winning strategy in the smaller game, it can be extended to a winning strategy in the larger game. Finally, if both players can force a draw in the smaller game, their respective strategies must force a draw in the larger game.

5.7 No-Draw, High-Low Poker

You can check that $0.75\text{RR} + 0.25\text{SR}, \text{SR}$ and $0.33\text{RR} + 0.67\text{RS}, \text{SR}$ are additional Nash equilibria.

Mixed-strategy Nash Equilibria: Answers

6.4 Tennis Strategy

$$\begin{aligned}
\sigma &= \alpha b_r + (1-\alpha) f_r, \quad \tau = \beta b_s + (1-\beta) f_s \\
\pi_{b_s} &= \alpha \pi_1(b_s, b_r) + (1-\alpha)\pi_1(b_s, f_r) \\
&\quad \text{where } \pi_1(b_s, b_r) = \text{the server's payoff to } b_s, b_r \\
&= .4\alpha + .7(1-\alpha) = .7 - .3\alpha \\
\pi_{f_s} &= \alpha \pi_1(f_s, b_r) + (1-\alpha)\pi_1(f_s, f_r) \\
&= .8\alpha + .1(1-\alpha) = .1 + .7\alpha \\
.7 - .3\alpha &= .1 + .7\alpha \Rightarrow \boxed{\alpha = 3/5} \\
\pi_{b_r} &= \beta \pi_2(b_s, b_r) + (1-\beta)\pi_2(f_s, b_r) \\
&= .6\beta + .2(1-\beta) = .2 + .4\beta \\
\pi_{f_r} &= \beta \pi_2(b_s, f_r) + (1-\beta)\pi_2(f_s, f_r) \\
&= .3\beta + .9(1-\beta) = .9 - .6\beta \\
.2 + .4\beta &= .9 - .6\beta \Rightarrow \boxed{\beta = 7/10}
\end{aligned}$$

Payoffs to Players:

$$\pi_1 = .4 \cdot \frac{3}{5} + .7 \cdot \frac{2}{5} = .52, \quad \pi_2 = .6 \cdot \frac{7}{10} + .2 \cdot \frac{3}{10} = .48.$$

6.8 Robin Hood and Little John

The payoff matrix is:

$$\begin{array}{c|c|c|}
 & G & W \\
\hline
G & \begin{array}{c} -\delta - \tau_{lj}/2 \\ -\delta - \tau_r/2 \end{array} & \begin{array}{c} 0 \\ -\tau_r \end{array} \\
\hline
W & \begin{array}{c} -\tau_{lj} \\ 0 \end{array} & \begin{array}{c} -\epsilon - \tau_{lj}/2 \\ -\epsilon - \tau_r/2 \end{array} \\
\hline
\end{array}$$

The pure Nash equilibria are:

$$\begin{aligned}
GG&: \quad \tau_r, \tau_{lj} \geq 2\delta \\
WG&: \quad 2\delta \geq \tau_{lj} \\
GW&: \quad 2\delta \geq \tau_r.
\end{aligned}$$

For the mixed-strategy equilibrium, we have

$$\alpha_{lj} = \frac{\epsilon + \tau_{lj}/2}{\epsilon + \delta}, \qquad \alpha_r = \frac{\epsilon + \tau_r/2}{\epsilon + \delta}$$

for $2\delta > \tau_r, \tau_{lj}$.

Suppose $\tau_r > \tau_{lj}$. Then, the socially optimal δ is any δ satisfying $\tau_r > 2\delta > \tau_{lj}$, because in this case it never pays to fight. The cost of crossing the bridge is τ_{lj} (or $\tau_r + 2\tau_{lj}$), including the crossing time itself. Of course, this makes Robin Hood wait all the time. He might prefer to lower or raise the costs of fighting. Will he? The payoff to the game to the players when $\tau_r > 2\delta > \tau_{lj}$ is $(-\tau_{lj}, 0)$.

Suppose Robin Hood can shift to lower-cost confrontation: we lower δ so $\tau_r > \tau_{lj} > 2\delta$. Then, GG is dominant, and the gain to the two players is $(-\delta - \tau_{lj}/2, -\delta - \tau_r/2)$, which is better for Robin Hood if and only if $-\tau_{lj} < -\delta - \tau_{lj}/2$, or $2\delta < \tau_{lj}$, *which is true*! Therefore, *Robin Hood gains if he can shift to a lower-cost form of fighting*.

Suppose Robin Hood can shift to a higher-cost warfare. We raise δ so $2\delta > \tau_r > \tau_{lj}$. Now the mixed-strategy solution obtains, and the payoff to Robin Hood is $(-\delta - \tau_{lj}/2)(\in +\tau_{lj}/2)/(\in +\delta)$, which it is easy to see is always less than $-\tau_{lj}$. Thus, *Robin Hood never wants to shift to a higher-cost form of fighting*, even though he would win some of the time.

6.9 The Motorist's Dilemma

The normal form matrix for the game is:

	G	W	C
G	$-\delta - \tau/2$,$-\delta - \tau/2$	0,$-\tau$	0,$-\tau$
W	$-\tau$,0	$-\epsilon - \tau/2$,$-\epsilon - \tau/2$	$-\tau$,0
C	$-\tau$,0	0,$-\tau$	$-\delta - \tau/2$,$-\delta - \tau/2$

Write $\sigma = \tau/2\delta < 1$, and let $u = (u_1, u_2)$ and $v = (v_1, v_2)$ represent Bob and Alice's mixed strategies, where (u_1, u_2) means play G with probability u_1, play W with probability u_2, and play C with probability $1 - u_1 - u_2$. Similarly for (v_1, v_2). Let $\delta = \{(x, y) | 0 \leq x, y, x + y \leq 1\}$, so δ is the strategy space for both players.

It is easy to check that the payoff to the pair of mixed strategies (u, v) for Bob is

$$\begin{aligned} f_1(u, v) = -(2\delta v_1 + (\delta + \tau/2)(v_2 - 1))u_1 - ((\delta - \tau/2)(v_1 - 1) \\ + (\delta + \epsilon)v_2)u_2 + (\delta - \tau/2)v_1 \\ + (\delta + \tau/2)(v_2 - 1), \end{aligned} \tag{A3}$$

and the payoff $f_2(u, v)$ to Alice is, by symmetry, $f_2(u, v) = f_1(v, u)$. The players reaction sets are given by

$$\begin{aligned} R_1 &= \{(u, v) \in \delta \times \delta | f_1(u, v) = \max_{\mu} f_1(\mu, v)\} \\ R_2 &= \{(u, v) \in \delta \times \delta | f_2(u, v) = \max_{\mu} f_2(\mu, v)\}, \end{aligned}$$

and the set of Nash equilibria is $R_1 \cap R_2$.

If the coefficients of u_1 and u_2 are negative in equation (A3), then (0,0) is the only best response for Bob.

6.11 Frankie and Johnny

Let π be the payoff to Johnny, and write $\overline{x} = (x_f + x_j)/2$. If $x_f < x_j$, then $y < \overline{x}$ implies $\pi = x_f$, and otherwise $\pi = x_j$. If $x_f > x_j$, then $y < \overline{x}$ implies $\pi = x_j$, and otherwise $\pi = x_f$. Since $\Pr\{y < \overline{x}\} = F(\overline{x})$, we have $\pi = x_f F(\overline{x}) + x_j(1 - F(\overline{x}))$ for $x_f \leq x_j$, and $\pi = x_j F(\overline{x}) + x_f(1 - F(\overline{x}))$ for $x_f > x_j$.

First, suppose $x_f < x_j$. The first-order conditions on x_f and x_j are then $\pi_{x_f} = F(\overline{x}) + f(\overline{x})(x_f - x_j)/2 = 0$, and $\pi_{x_j} = 1 - F(\overline{x}) + f(\overline{x})(x_f - x_j)/2 = 0$, from which it follows that $F(\overline{x}) = 1/2$. Substituting into the first-order conditions gives $x_f = \overline{x} - 1/2f(\overline{x})$, $x_j = \overline{x} + 1/2f(\overline{x})$. Since π should be a minimum for Frankie, the second order condition must satisfy $\pi_{x_f x_f} = f(\overline{x}) + f'(\overline{x})(x_j - x_f)/4 > 0$. Since π should be a maximum for Johnny, the second order condition must satisfy $\pi_{x_j x_j} = -f(\overline{x}) + f'(\overline{x})(x_j - x_f)/4 < 0$.

For instance, if y is drawn from a uniform distribution then $\overline{x} = 1/2$ and $f(\overline{x}) = 1$, so $x_f = 0$ and $x_j = 1$. For another example, suppose $f(x)$ is quadratic, symmetric about $x = 1/2$, and $f(0) = f(1) = 0$. Then it is easy to check that $f(x) = 6x(1 - x)$. In this case $\overline{x} = 1/2$ and $f(\overline{x}) = 3/2$, so $x_f = 1/6$ and $x_j = 5/6$.

6.13 Cheater-Inspector

Let α be the probability of trusting. If there is a mixed-strategy equilibrium in the n-round game, the payoff to cheating in the first period is $\alpha n + (1-\alpha)(-an) = \alpha n(1+a) - an$, and the payoff to being honest is $g_{n-1} + b(1-\alpha)$. Equating these, we find

$$\alpha = \frac{g_{n-1} + b + an}{n(1+a) + b},$$

assuming $g_{n-1} < n$ (which is true for $n = 0$, and which we will show is true for larger n by induction). The payoff of the n-round game is then

$$g_n = g_{n-1} + b\frac{n - g_{n-1}}{n(1+a) + b}.$$

It is easy to check that $g_1 = b/(1+a+b)$ and $g_2 = 2b/(1+a+b)$, which suggests that

$$g_n = \frac{nb}{1+a+b}.$$

This can be checked directly by assuming it to be true for g_{n-1} and proving it true for g_n. This is called "proof by induction": prove it for $n = 1$, then show that it is true for some integer n, it is true for $n+1$. Then it is true for all integers n

$$\begin{aligned} g_n &= g_{n-1} + b\frac{n - g_{n-1}}{n(1+a) + b} \\ &= \frac{b(n-1)}{1+a+b} + b\frac{n - \frac{b(n-1)}{1+a+b}}{n(1+a) + b} \\ &= \frac{b(n-1)}{1+a+b} + \frac{b}{1+a+b}\frac{n + na + nb - b(n-1)}{n(1+a) + b} \\ &= \frac{b(n-1)}{1+a+b} + \frac{b}{1+a+b} \\ &= \frac{bn}{1+a+b}. \end{aligned}$$

6.16 Big John and Little John Revisited

Let σ be the mixed strategy for Big John, who climbs with probability α, and let τ be the strategy for Little John, who climbs with probability β. Let π_{c_i} and π_{w_i} be the payoffs to climbing and waiting, respectively, for player i. Then we have

$$
\begin{aligned}
\sigma &= \alpha c_1 + (1-\alpha)w_1, \quad \tau = \beta c_2 + (1-\beta)w_2 \\
\pi_{c_1} &= \beta\pi_1(c_1,c_2) + (1-\beta)\pi_1(c_1,w_2) \\
&\quad \text{where } \pi_1(c_1,c_2) = \text{Big John's payoff to } c_1, c_2 \\
&= 5\beta + 4(1-\beta) = 4+\beta \\
\pi_{w_1} &= \beta\pi_1(w_1,c_2) + (1-\beta)\pi_1(w_1,w_2) \\
&= 9\beta + 0(1-\beta) = 9\beta \\
4+\beta &= 9\beta \Rightarrow \boxed{\beta = 1/2} \\
\pi_{c_2} &= \alpha\pi_2(c_1,c_2) + (1-\alpha)\pi_2(w_1,c_2) \\
&= 3\alpha + (1-\alpha) = 1+2\alpha \\
\pi_{w_2} &= \alpha\pi_2(c_1,w_2) + (1-\alpha)\pi_2(w_1,w_2) \\
&= 4\alpha + 0(1-\alpha) = 4\alpha \\
1+2\alpha &= 4\alpha \Rightarrow \boxed{\alpha = 1/2}.
\end{aligned}
$$

Payoffs to Players:

$$\pi_1 = 5\cdot\frac{1}{2} + 4\cdot\frac{1}{2} = \frac{9}{2}, \quad \pi_2 = 3\cdot\frac{1}{2} + 1\cdot\frac{1}{2} = 2.$$

Note: Show two other ways to find payoffs

6.21 One-Card, Two-Round Poker with Bluffing

The reduced normal form is as follows:

	ss	sf	f
rrbb	0,0	4,–4	2,–2
rrbf	1,–1	0,0	2,–2
rrf	2,–2	1,–1	0,0
rfbb	–5,5	0,0	2,–2
rfbf	–4,4	4,–4	2,–2
rff	–3,3	–3,3	0,0
fbb	–4,4	1,–1	0,0
fbf	–3,3	–3,3	0,0
ff	–2,2	–2,2	–2,2

The last six strategies for player 1 are weakly dominated by rrbb. Eliminating these strategies gives the following reduced normal form.

	ss	sf	f
rrbb	0,0	4,–4	2,–2
rrbf	1,–1	0,0	2,–2
rrf	2,–2	1,–1	0,0

If 2 uses α ss + β sf + $(1-\alpha-\beta)$ f, the payoffs to 1's strategies are:

$$\begin{aligned} \text{rrbb:}&\quad 4\beta + 2(1-\alpha-\beta) = -2\alpha + 2\beta + 2 \\ \text{rrbf:}&\quad \alpha + 2(1-\alpha-\beta) = -\alpha - 2\beta + 2 \\ \text{rrf:}&\quad 2\alpha + \beta \end{aligned}$$

If rrbb and rrbf are used, we have $\beta = \alpha/4$; if rrbb and rrf are used, we have $4\alpha = \beta + 2$. If rrbf and rrf are used we have $\alpha + \beta = 2/3$. Thus, if all three are used, we have $\alpha = 8/15, \beta = 2/15$, and $1 - \alpha - \beta = 1/3$. The payoff is 18/15 = 6/5.

If 1 uses γ rrbb + δ rrbf + $(1-\gamma-\delta)$ rrf, the payoffs to 2's strategies are

$$\begin{aligned} \text{ss:}&\quad -\delta - 2(1-\gamma-\delta) = 2\gamma + \delta - 2 \\ \text{sf:}&\quad -4\gamma - (1-\gamma-\delta) = -3\gamma + \delta - 1 \\ \text{f:}&\quad -2\gamma - 2\delta \end{aligned}$$

Thus, if ss and sf are used, $\gamma = 1/5$. If ss and f are both used, $4\gamma + 3\delta = 2$, so if all are used, $3\delta = 2 - 4/5 = 6/5$, and $\delta = 2/5$. Then $1 - \gamma - \delta = 2/5$. The payoff is $4/5 - 2 = -1/5 - 1 = -6/5$, so it all works out.

There is a Nash equilibrium

$$\frac{8}{15}\text{ss} + \frac{2}{15}\text{sf} + \frac{1}{3}\text{f}, \quad \frac{1}{5}\text{rrbb} + \frac{2}{5}\text{rrbf} + \frac{2}{5}\text{rrf},$$

with a payoff of 6/5 to player 1.

Note that we have arrived at this solution by eliminating weakly dominated strategies. Have we eliminated any Nash equilibria this way?

6.23 Trust in Networks

Let α and β be the fraction of inspectors and trusters, respectively, and write $\gamma = 1 - \alpha - \beta$. Then we have

$$\begin{aligned}\pi_I &= \alpha\pi_{II} + \beta\pi_{IT} + \gamma\pi_{ID}\\ \pi_T &= \alpha\pi_{TI} + \beta\pi_{TT} + \gamma\pi_{TD}\\ \pi_D &= \alpha\pi_{DI} + \beta\pi_{DT} + \gamma\pi_{DD}\end{aligned}$$

where $\pi_{II} = p^2, \pi_{IT} = p, \pi_{ID} = -2(1-p), \pi_{TI} = p, \pi_{TT} = 1, \pi_{TD} = -2, \pi_{DI} = 2(1-p), \pi_{DT} = 2$, and $\pi_{DD} = -1$. Solving simultaneously for the completely mixed Nash equilibrium, we find $\pi^*(p) = 4(1-2p)^2/(1+p)(3p-1)$, which has derivative $8(1-7p+10p^2)/(1-3p)^2(1+p)^2$, which is positive for $p \in (0.5, 1]$.

6.27 A Mating Game

Let $\alpha = \alpha_H + \alpha_E$, and $\beta = \beta_H + \beta_E$. You can check that the payoffs for males are (a) $\pi^m_{FF} = 1$; (b) $\pi^m_{FR} = 3(2-\alpha)/4$; (c) $\pi^m_{RF} = 3\alpha/4$; (d) $\pi^m_{RR} = 1$. The payoffs for females are (a) $\pi^f_{FF} = 1-\beta/4$; (b) $\pi^f_{FR} = 2-\beta$; (c) $\pi^f_{RF} = \beta/2$; (d) $\pi^f_{RR} = 1-\beta/4$. Also, $\alpha, \beta = 2$ for FF, $\alpha, \beta = 1$ for FR and RF, and $\alpha, \beta = 0$ for RR. Now you can form the 4×4 normal form matrix, and the rest is straightforward.

6.28 Coordination Failure

Suppose player 1 uses the three pure strategies with probabilities α,β, and $\gamma = 1-\alpha-\beta$, respectively, and 2 uses the pure strategies with probabilities

a, b, and $c = 1 - a - b$. We can assume without loss of generality that $\alpha \geq 1/3$ and $\beta \geq \gamma$. The payoffs to a, b, and c are

$$\begin{aligned}\pi_a &= 50\beta + 40(1-\alpha-\beta) = 40 - 40\alpha + 10\beta,\\ \pi_b &= 40\alpha + 50(1-\alpha-\beta) = 50 - 10\alpha - 50\beta,\\ \pi_c &= 50\alpha + 40\beta.\end{aligned}$$

We have $\alpha + 2\beta \geq 1$, so $\beta \geq (1-\alpha)/2$. Then,

$$\begin{aligned}\pi_c - \pi_a &= 50\alpha + 40\beta - [40 - 40\alpha + 10\beta] = 90\alpha + 30\beta - 40\\ &> 90\alpha + 30(1-\alpha)/2 - 40 = 15 + 75\alpha - 40 = 75\alpha - 25 > 0.\end{aligned}$$

Thus, c is better than a. Also,

$$\begin{aligned}\pi_c - \pi_b &= 50\alpha + 40\beta - [50 - 10\alpha - 50\beta] = 60\alpha + 90\beta - 50\\ &> 60\alpha + 90(1-\alpha)/2 - 50 = 45 + 15\alpha - 50 = 15\alpha - 5 > 0,\end{aligned}$$

so c is better than b. Thus, player 2 will use c, and his payoff is $50\alpha + 40\beta > 50\alpha + 20(1-\alpha) = 20 + 30\alpha > 30$. The payoff to 1 is then $40\alpha + 50\beta > 40\alpha + 25(1-\alpha) = 25 + 15\alpha > 30$. Thus, both are better off than with the 30 payoff of the Nash equilibrium.

6.29 Colonel Blotto Game

The payoff matrix, giving Colonel Blotto's return (the enemy's payoff is the negative of this) is as follows:

		Enemy Strategies			
		(3,0)	(0,3)	(2,1)	(1,2)
	(4,0)	4	0	2	1
Colonel Blotto Strategies	(0,4)	0	4	1	2
	(3,1)	1	−1	3	0
	(1,3)	−1	1	0	3
	(2,2)	−2	−2	2	2

Suppose the enemy uses all strategies. By symmetry, 1 and 2 must be used equally, and 3 and 4 must be used equally. Let p be the probability of using (3,0), and q be the probability of using (2,1). The expected return to Colonel Blotto is then

$$\begin{array}{rrrrcr}
4p+ & & 2q+ & q= & & 4p+3q \\
 & 4p+ & q & +2q= & & 4p+3q \\
p- & p & +3q & & = & 3q \\
-p+ & p & & +3q= & & 3q \\
-2p & -2p & +2q & +2q & = & -4p+4q.
\end{array}$$

Colonel Blotto cannot use all strategies in a mixed strategy, because there is no p that makes all entries in this vector equal. Suppose we drop Colonel Blotto's (3,1) and (1,3) strategies and choose p to solve $4p+3q=-4p+4q$ and $2p+2q=1$. Thus, $p=1/18$ and $q=4/9$. There are other Nash equilibria.

6.30 Number Guessing Game

Clearly, the game is determined in the first two rounds. Let us write my strategies as (g h l), for "first guess g, if high guess h and if low guess l." If a high guess is impossible, we write (1 x l), and if a low guess is impossible, we write (3 h x). For instance, (1x3) means "first choose 1, and if this is low, then choose 3." Then, we have the following payoff matrix for Bob (the payoff to Alice is minus the payoff to Bob):

	Alice				
Bob	(1x2)	(1x3)	(213)	(31x)	(32x)
1	1	1	2	2	3
2	2	3	1	3	2
3	3	2	2	1	1

First show that Bob will use a completely mixed strategy. It is obvious that no Nash equilibrium uses only a single pure strategy of Alice, and you can show that no Nash equilibrium uses one of the 10 pairs of pure strategies for Alice. This leaves the 10 triples of strategies for Alice. It is arduous to check all of these, but the procedures described in section 6.44. It turns out that there is only one Nash equilibrium, in which we drop (1x2) and (32x). Then, equating the costs of the other three, we find that Bob uses the mixed strategy (0.4,0.2,0.4) against Alice's mixed strategy (0.2,0.6,0.2). The payoff is 1.8 to player 1. It is easy to check that Alice's excluded strategies are more costly than this for Alice.

6.31 Target Selection

Suppose Attacker uses mixed strategy $x = (x_1, \dots, x_n)$ and Defender uses strategy $y = (y_1, \dots, y_n)$, and these form a Nash equilibrium. If $x_j = 0$, then the best response of Defender must set $y_j = 0$. Suppose $x_i > 0$ for some $i > j$. Then, by switching x_i and x_j, Attacker gains $a_j - pa_i y_i \geq a_j - a_i > 0$.

6.32 A Reconnaissance Game

The normal form matrix is as follows:

	counter full defend	counter half defend	no counter full defend	no counter half defend
reconnoiter, full attack	$a_{11} - c + d$	$a_{12} - c + d$	$a_{11} - c$	$a_{12} - c$
reconnoiter, half attack	$a_{21} - c + d$	$a_{22} - c + d$	$a_{21} - c$	$a_{22} - c$
no reconnoiter, full attack	$a_{11} + d$	$a_{12} + d$	a_{11}	a_{12}
no reconnoiter, half attack	$a_{21} + d$	$a_{22} + d$	a_{21}	a_{22}

With the given payoffs and costs, the entries in the normal form game become

46,–46	22,–22	39,–39	27,–27
10,–10	34,–34	39,–39	27,–27
55,–55	31,–31	48,–48	24,–24
19,–19	43,–43	12,–12	36,–36

Suppose Defender does not counter and full defends with probability p. Then, Attacker faces

$$\begin{aligned} 39p + 27(1-p) &= 12p + 27 \\ 39p + 27(1-p) &= 12p + 27 \\ 48p + 24(1-p) &= 24p + 24 \\ 12p + 36(1-p) &= -24p + 36. \end{aligned}$$

Check the third and fourth. We have $-24p + 36 = 24p + 24$, so $p = 1/4$. Suppose attacker does not reconnoiter and full attacks with probability q. Then, $-48q - 12(1-q) = -24q - 36(1-q)$, so $q = 1/2$. You must check that no other strategy has a higher payoff, and you will find this to be true. The payoffs are $(30, -30)$. If you are ambitious, you can check that there are many other Nash equilibria, all of which involve (0,0,1/4,3/4) for Defender. How do you interpret this fact?

6.33 Attack on Hidden Object

We have

	P	F
PP	$2\gamma - \gamma^2$	$\beta\gamma$
PF	γ	γ
FP	$\beta(1-\alpha(1-\beta))$	β
FF	β^2	$2\beta - \beta^2$

Note that the second row is strongly dominated by the third, and the third row is weakly dominated by the fourth row. Moreover, it is clear that if $\beta^2 > 2\gamma - \gamma^2$, then the first row is strictly dominated by the fourth, so there is a unique Nash equilibrium in which Bob plays FF and Alice plays P. The condition for this is

$$\alpha > \frac{\beta + \sqrt{1-\beta^2} - 1}{\beta}.$$

If $\beta^2 < 2\gamma - \gamma^2 < \beta(1-\alpha(1-\beta))$, then PP is strictly dominated by FP, and (FP,P) is a pure-strategy equilibrium. Finally, if $2\gamma - \gamma^2 > b(1-\alpha(1-\beta))$ you can check that there is a strictly mixed Nash equilibrium including PP, FP for Bob, and F and P for Alice.

Principal-Agent Models: Answers

7.1 Gift Exchange

Choosing w and N to maximize profits gives the first-order conditions

$$\pi_w(w,N) = [f'(eN)e' - 1]N = 0 \tag{A4}$$
$$\pi_N(w,N) = f'(eN)e - w = 0. \tag{A5}$$

Solving these equations gives the Solow condition.

The second partials are

$$\pi_{ww} = [f''Ne'^2 + f'e'']N < 0, \qquad \pi_{NN} = f''e^2 < 0,$$
$$\pi_{wN} = f''Nee' + f'e' - 1 = f''Nee' < 0.$$

It is easy to check that the second-order conditions are satisfied: $\pi_{ww} < 0$, $\pi_{NN} < 0$, and $\pi_{ww}\pi_{NN} - \pi_{wN}^2 > 0$.

To show that $dw/dz > 1$, differentiate the first-order conditions (AA5) totally with respect to w and N:

$$\begin{aligned} \pi_{ww}\frac{dw}{dz} + \pi_{wN}\frac{dN}{dz} + \pi_{wz} &= 0 \\ \pi_{Nw}\frac{dw}{dz} + \pi_{NN}\frac{dN}{dz} + \pi_{Nz} &= 0. \end{aligned} \tag{A6}$$

Solving these two equations in the two unknowns dw/dz and dN/dz, we find

$$\frac{dw}{dz} = -\frac{\pi_{NN}\pi_{wz} - \pi_{Nw}\pi_{Nz}}{\pi_{NN}\pi_{wz} - \pi_{Nw}^2}. \tag{A7}$$

But we also calculate directly that

$$\pi_{wz} = -[f''Ne'^2 + f'e'']N = -\pi_{ww},$$
$$\pi_{Nz} = -f'e' - f''Nee' = -f'e' - \pi_{Nw}.$$

Substituting these values in (A7), we get

$$\frac{dw}{dz} = 1 - \frac{\pi_{Nw}f'e'}{\pi_{NN}\pi_{wz} - \pi_{Nw}^2},$$

and the fraction in this expression is negative (the denominator is positive by the second-order conditions, while $\pi_{Nw} < 0$ and $f', e' > 0$).

Because $dw/dz > 1$, it follows from the chain rule that

$$\begin{aligned}
\frac{de}{dz} &= e'\left[\frac{dw}{dz} - 1\right] > 0, \\
\frac{dN}{dz} &= \frac{-\pi_{wz} - \pi_{ww}\frac{dw}{dz}}{\pi_{wN}} \\
&= \frac{\pi_{ww}}{\pi_{wN}}\left(1 - \frac{dw}{dz}\right) < 0 \qquad \text{[by (AA6), (AA7)]}, \\
\frac{d\pi}{dz} &= \frac{\partial\pi}{\partial w}\frac{dw}{dz} + \frac{\partial\pi}{\partial N}\frac{dN}{dz} + \frac{\partial\pi}{\partial z} = \frac{\partial\pi}{\partial z} = -f'e'N < 0.
\end{aligned}$$

7.6 Bob's Car Insurance

Suppose Bob is careful without insurance, so we know $\epsilon \leq 0.177$. Because the insurance company's lottery is fair, we have $x = 0.95(1200-z)$ if Bob is careful with insurance, and $x = 0.925(1200-z)$ if Bob is careless with insurance. We cannot assume he will be careless in this case, because the deductible might induce Bob to be careful.

If Bob is careless with insurance, the value of car plus insurance is $v = 0.925\ln(1201-x) + 0.075\ln(1201-z-x)$, and because the insurance is fair, we have $x=0.075(1200-z)$. Subsituting the second expression for x in the first, and taking the derivative with respect to z, we find

$$\frac{dv}{dz} \approx \frac{z}{z^2 + 13612Z - 17792000},$$

which is negative for $z \in (0, 1200)$. Thus, zero deductible is optimal.

Now suppose Bob is careful. Then, the value of car plus insurance is $v = 0.95\ln(1201-x) + 0.05\ln(1201-z-x)$, and fair insurance implies $x=0.05(1200-z)$. The derivative of this with respect to z is

$$\frac{dv}{dz} \approx \frac{z}{z^2 + 21618.9Z - 27408000},$$

which is also negative, so the optimal deductible for Bob is zero. However, in this case, z must be sufficiently large that Bob wants to be careful, or the insurance company will not be willing to issue the insurance at the low rate $x = 0.05(1200 - z)$. To make taking care worthwhile, we must have

$$\begin{aligned}&0.95\ln(1201 - z) + 0.05\ln(1201 - z - x) - \epsilon \geq\\&\quad 0.925\ln(1201 - z) + 0.075\ln(1201 - z - x).\end{aligned}$$

The minimum z satisfying this is when the equality holds. This equation cannot be solved analytically, but calculations show that there is no solution for $\epsilon > 0.0012$, and when $\epsilon = 0.001$, the deductible must be at least $z =$ \$625.

Signaling Games: Answers

8.3 Introductory Offers

If a high-quality firm sells to a consumer in the first period at some price p_1, then in the second period the consumer will be willing to pay $p_2 = h$, because he knows the product is of high quality. Knowing that it can make a profit $h - c_h$ from a customer in the second period, a high-quality firm might want to make a consumer an "introductory offer" at a price p_1 in the first period that would not be mimicked by the low-quality firm, in order to reap the second-period profit.

If $p_1 > c_l$, the low-quality firm could mimic the high-quality firm, so the best the high-quality firm can do is to charge $p_1 = c_l$, which the low-quality firm will not mimic, because the low-quality firm cannot profit by doing so (it cannot profit in the first period, and the consumer will not buy the low-quality product in the second period). In this case, the high-quality firm's profits are $(c_l - c_h) + \delta(h - c_h)$. As long as these profits are positive, which reduces to $h > c_h + \delta(c_h - c_l)$, the high-quality firm will stay in business.

8.6 The Shepherds Who Never Cry Wolf

The following payoffs are easy to derive:

$$\pi_1(N,N) = p(1-a) + (1-p)(1-b); \quad \pi_2(N,N) = 1;$$
$$\pi_1(N,H) = p(1-a) + (1-p)(1-b); \quad \pi_2(N,H) = 1;$$
$$\pi_1(N,A) = 1; \quad \pi_2(N,A) = 1-d;$$
$$\pi_1(H,N) = p(1-a) + (1-p)(1-b) - pc; \quad \pi_2(H,N) = 1;$$
$$\pi_1(H,H) = p(1-c) + (1-p)(1-b); \quad \pi_2(H,H) = p(1-d) + 1 - p;$$
$$\pi_1(H,A) = 1 - pc; \quad \pi_2(H,A) = 1-d;$$
$$\pi_1(A,N) = p(1-a) + (1-p)(1-b) - c; \quad \pi_2(A,N) = 1;$$
$$\pi_1(A,H) = 1-c; \quad \pi_2(A,H) = 1-d;$$
$$\pi_1(A,A) = 1-c; \quad \pi_2(A,A) = 1-d.$$

Now the total payoff for shepherd 1 is $\pi_1^t = \pi_1 + k\pi_2$, and the total payoff for shepherd 2 is $\pi_2^t = \pi_1 + k\pi_1$. Substituting in numbers and forming the normal form matrix for the game, we get

	N	H	A
N	$\frac{19}{24}, \frac{37}{32}$	$\frac{19}{24}, \frac{37}{32}$	$\frac{21}{16}, \frac{7}{6}$
H	$\frac{95}{192}, \frac{793}{768}$	$\frac{47}{48}, \frac{829}{768}$	$\frac{65}{64}, \frac{267}{256}$
A	$\frac{19}{48}, \frac{571}{576}$	$\frac{11}{12}, \frac{577}{768}$	$\frac{11}{12}, \frac{577}{576}$

It is easy to see that (H, H) and (N, A) are Nash equilibria, and you can check that there is a mixed-strategy equilibrium in which the threatened shepherd uses $\frac{1}{3}N + \frac{2}{3}H$ and the other shepherd uses $\frac{3}{5}H + \frac{2}{5}A$.

8.8 Honest Signaling among Partial Altruists

The payoff matrix for the encounter between a fisher observing a threatened fisher is as follows, where the first two lines are the payoffs to the individual players, and the third is the total payoff:

	Never Ask	Ask If Distressed	Always Ask
Never Help	$r(1-p)u$ $(1-p)u$ $(1+r)(1-p)u$	$r(1-p)u-rpt$ $(1-p)u-pt$ $(1+r)[(1-p)u-pt]$	$r(1-p)u-rt$ $(1-p)u-t$ $(1+r)[(1-p)u-t]$
Help If Asked	$r(1-p)u$ $(1-p)u$ $(1+r)(1-p)u$	$r[p(1-t)+(1-p)u]-pc$ $p(1-t)+(1-p)u$ $(1+r)[p(1-t)+(1-p)u-pc]$	$r[p+(1-p)v-t]-c$ $p+(1-p)v-t$ $(1+r)[p+(1-p)v-t]-c$
Always Help	$r[p+(1-p)v]-c$ $p+(1-p)v$ $(1+r)[p+(1-p)v]-c$	$r[p(1-t)+(1-p)v]-c$ $p(1-t)+(1-p)v$ $(1+r)[p(1-t)+(1-p)v]-c$	$r[p+(1-p)v-t]-c$ $p+(1-p)v-t$ $(1+r)[p+(1-p)v-t]-c$

The answers to the problem can be obtained in a straightforward manner from this matrix.

8.10 Education as a Screening Device

a. Given the probabilities (c), the wages (b) follow from

$$w_k = \mathrm{P}[a_h|e_k]a_h + \mathrm{P}[a_l|e_k]a_l, \qquad k = h, l. \tag{A8}$$

Then, it is a best response for workers to choose low education whatever their ability type, so (a) follows. Because both types choose e_l, the conditional probability $\mathrm{P}[a_l|e_l] = 1-\alpha$ is consistent with the behavior of the agents, and because e_h is off the path of play, any conditional for $\mathrm{P}[a_l|e_h]$ is acceptable, so long as it induces a Nash equilibrium.

b. Assume the above conditions hold, and suppose c satisfies $a_l(a_h-a_l) < c < a_h(a_h-a_l)$. The wage conditions (b) follow from (A8) and (c). Also, $a_h - c/a_h > a_l$, so a high-ability worker prefers to choose $e = 1$ and signal his true type, rather than choose e_l and signal his type as low ability. Similarly, $a_l > a_h - c/a_l$, so a low-ability worker prefers to choose e_l and signal his true type, rather than choose e_h and signal his type as high ability.

c. The wage conditions (b) follow from (A8) and (c). Suppose $c < a_l(a_h - a_l)$. Then both high- and low-ability workers prefer to get education and the higher wage w_h rather than signal that they are low quality.

d. Let $\bar{e} = \alpha a_l(a_h - a_l)/c$, and choose $e^* \in [0, \bar{e}]$. Given the employer's wage offer, if a worker does not choose $e = e^*$ he might as well choose $e = 0$, because his wage in any case must be $w = a_l$. A low-ability worker then prefers to get education e^* rather than any other educational level, because $a_l \leq \alpha a_h + (1-\alpha)a_l - ce^*/a_l$. This is thus true for the high-ability worker, whose incentive compatibility constraint is not binding.

e. Consider the interval

$$\left[\frac{a_l(a_h-a_l)}{c}, \frac{a_h(a_h-a_l)}{c}\right].$$

If c is sufficiently large, this interval has a nonempty intersection with the unit interval $[0,1]$. Suppose this intersection is $[e_{min}, e_{max}]$. Then, for $e^* \in [e_{min}, e_{max}]$, a high-ability worker prefers to acquire education e^* and receive the high wage $w = a_h$, whereas the low-ability worker prefers to receive $w = a_l$ with no education.

8.11 Capital as a Signaling Device

a. Given $p > 0$, choose k so that

$$1 > k(1+\rho) > q + p(1-q).$$

This is possible because $q + p(1-q) < 1$. Then it is clear that the fraction q of good projects is socially productive. The interest rate r that a producer must offer then must satisfy

$$k(1+\rho) = qk(1+r) + (1-q)kp(1+r) = k(1+r)[q + p(1-q)],$$

so

$$r = \frac{1+\rho}{q + p(1-q)} - 1. \tag{A9}$$

The net profit of a producer with a good project is then

$$1 - k(1+r) = \frac{q + p(1-q) - k(1+\rho)}{q + p(1-q)} < 0,$$

so such producers will be unwilling to offer lenders an interest rate they are willing to accept. The same is clearly true of bad projects, so no projects get funded. Note that bad projects are not socially productive in this case, because $p - k(1+\rho) < p - (q + p(1-q)) = -q(1-p) < 0$.

b. Choose k so that $p < k(1+\rho) < q + p(1-q)$, which is clearly always possible. Then the fraction $1-q$ of bad projects are socially unproductive. The interest rate r must still satisfy equation (A9), so the payoff to a successful project (good or bad) is

$$1 - k(1+r) = \frac{q + p(1-q) - k(1+\rho)}{q + p(1-q)} > 0,$$

so producers of both good and bad projects are willing to offer interest rate r, and lenders are willing to lend at this rate to all producers.

c. Let

$$k^p_{min} = \frac{p[1 - k(1+\rho)]}{(1-p)(1+\rho)}.$$

Note that because good projects are socially productive, $k^p_{min} > 0$. Suppose all producers have wealth $k^p > k^p_{min}$, and lenders believe that only a producer with a good project will invest k^p in his project. Then lenders will be willing to lend at interest rate ρ. If a producer invests k^p in his project and borrows $k - k^p$, his return is 1 and his costs are forgone earnings $k^p(1+\rho)$ and capital costs $(k - k^p)(1+\rho)$. Thus, his profit is

$$1 - k^p(1+\rho) - (k - k^p)(1+\rho) = 1 - k(1+\rho) > 0,$$

so such a producer is willing to undertake this transaction. If the producer with a bad project invests his capital k^p, his return is

$$p[1 - k(1 + \rho)] - (1 - p)k^p(1 + \rho) < 0,$$

so he will not put up the equity. This proves the theorem.

Repeated Games : Answers

9.4 The Strategy of an Oil Cartel

Payoffs:

Low/Low	$(25-2)\times 2, (25-4)\times 2$	=	46,42
High/Low	$(15-2)\times 4, (15-4)\times 2$	=	52,22
Low/High	$(15-2)\times 2, (15-4)\times 4$	=	26,44
High/High	$(10-2)\times 4, (10-4)\times 4$	=	32,24

Normal Form Game:

	Low	High
Low	46,42	26,44
High	52,22	32,24

The condition for the cooperate payoff to be higher than the defect payoff for Iran is

$$\frac{46}{1-\delta} > 52 + \delta\frac{32}{1-\delta}.$$

We can solve this, getting $\delta > 0.3$, which corresponds to an interest rate r given by $r = (1-\delta)/\delta = 0.7/0.3 \approx 233.33\%$. The condition for cooperate to beat defect for Iraq is

$$\frac{42}{1-\delta} > 44 + \delta\frac{24}{1-\delta}.$$

We can solve this, getting $\delta > 0.1$, which corresponds to an interest rate r given by $r = 900\%$.

9.5 Reputational Equilibrium

If it is worthwhile for the firm to lie when it claims its product has quality $q > 0$, it might as well set its actual quality to 0, because the firm minimizes costs this way. Its profits are then

$$\pi_f = (4 + 6q_a - x - 2)x = (2 + 6q_a - x)x.$$

Profits are maximized when

$$\frac{d\pi_f}{dx} = 2 + 6q_a - 2x = 0,$$

so $x = 1 + 3q_a$, and $\pi_f = (1 + 3q_a)^2$.

Now suppose the firm tells the truth. Then, if π_t is per-period profits, we have

$$\pi_t = (2 + 6q_a - 6q_a^2 - x)x,$$
$$\frac{d\pi_t}{dx} = 2 + 6q_a - 6q_a^2 - 2x = 0,$$

so $x = 1 + 3q_a - 3q_a^2$, and $\pi_t = (1 + 3q_a - 3q_a^2)^2$. But total profits Π from truth telling are π_t forever, discounted at rate $\delta = 0.9$, or

$$\Pi = \frac{\pi_t}{1-\delta} = 10(1 + 3q_a - 3q_a^2)^2.$$

Truth-telling is profitable then when $\Pi \geq \pi_f$, or when

$$10(1 + 3q_a - 3q_a^2)^2 > (1 + 3q_a)^2. \tag{A10}$$

Note that equation (A10) is true for very small q_a (that is, q_a near 0) and false for very large q_a (that is, q_a near 1).

Evolutionary Stable Strategies: Answers

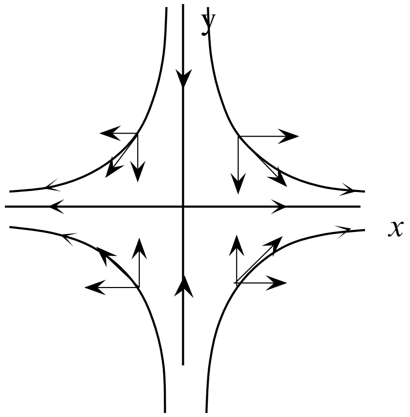
10.4 A Symmetric Coordination Game

a. Let s_a and s_b be the two strategies, and write $\sigma = \alpha s_a + (1-\alpha)s_b$ for the mixed strategy where s_a is played with probability α. If $\tau = \beta s_a + (1-\beta)s_b$, we have $\pi[\sigma,\tau] = \alpha\beta a + (1-\alpha)(1-\beta)b$. Suppose (σ,σ) is a Nash equilibrium. Then by the fundamental theorem (§3.6), $\pi[s_a,\sigma] = \pi[s_b,\sigma]$, which implies $\alpha = b/(a+b)$. Note that $\pi[\sigma,\sigma] = ab/(a+b)$, which is smaller than either a or b. We shall show that b can invade a population that plays σ. By the fundamental theorem, $\pi[s_b,\sigma] = \pi[\sigma,\sigma]$, because $\alpha < 1$. Thus σ is impervious to invasion by s_b only if $\pi[\sigma,s_b] > \pi[s_b,s_b]$, which reduces to $ab/(a+b) > b$, which is false.

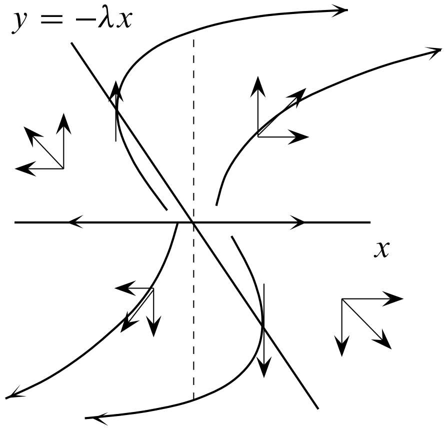
Dynamical Systems: Answers

11.10 Exercises in Two-Dimensional Linear Systems

(b)



(d)



Evolutionary Dynamics: Answers

12.5 Properties of the Replicator System

Only the last part of the question might not be obvious. Let $p(t)$ be a trajectory of (12.5) and define

$$b(t) = \int_0^t \frac{dt}{a(p(t),t)},$$

which is possible because $a(p,t) > 0$. Clearly, $b(t)$ is positive and increasing. Let $q(t) = p(b(t))$. Then, by the fundamental theorem of the calculus,

$$\begin{aligned}\dot{q}_i(t) &= \dot{b}(t)\dot{p}_i(b(t)) = \frac{1}{a(t)}a(t)p_i(b(t))(\pi_i(p(b(t))) - \bar{\pi}(p(b(t))))\\ &= q_i(t)(\pi_i(q(t)) - \bar{\pi}(q(t))). \blacksquare\end{aligned}$$

12.10 Trust in Networks III

You can check that the eigenvalues of the Jacobian at the equilibrium are given by

$$\begin{aligned}\lambda_1, \lambda_2 = {} & \frac{-2 + 5p - 4p^2 + p^3}{2(1+p)(3p-1)}\\ & \pm \frac{\sqrt{4 - 60p + 177p^2 - 116p^4 - 110p^4 + 104p^5 + p^6}}{2(1+p)(3p-1)}.\end{aligned}$$

This is pretty complicated, but you can check that the expression under the radical is negative for p near unity: factor out $(p-1)$ and show that the other factor has value 32 when $p = 1$. The rest of the expression is real and negative for p near unity, so the equilibrium is a stable focus.

12.13 A Generalization of Rock, Paper, and Scissors

Note first that no pure strategy is Nash. If one player randomizes between two pure strategies, the other can avoid the -1 payoff, so only strictly mixed solutions can be Nash. Check that the only such strategy σ that is Nash uses probabilities (1/3,1/3,1/3). This is not evolutionarily stable for $\alpha < 0$, however, because the pure strategy R has payoff $\alpha/3$ against σ, which is also the payoff to σ against σ, and has payoff α against itself.

The payoff of the strategies against $(x_1, x_2, 1 - x_1 - x_2)$ are

$$\begin{aligned}
R\colon &\quad \alpha x_1 + x_2 - (1 - x_1 - x_2) &=& \quad (1+\alpha)x_1 + 2x_2 - 1\\
P\colon &\quad -x_1 + \alpha x_2 + (1 - x_1 - x_2) &=& \quad -2x_1 - (1-\alpha)x_2 + 1\\
S\colon &\quad x_1 - x_2 + \alpha(1 - x_1 - x_2) &=& \quad (1-\alpha)x_1 - (\alpha+1)x_2 + \alpha
\end{aligned}$$

The average payoff is then $2\alpha(x_1^2 + x_1x_2 + x_2^2 - x_1 - x_2) + \alpha$, and the fitnesses of the three types are

$$\begin{aligned}
f_1\colon &\quad (1+3\alpha)x_1 + 2(1+\alpha)x_2 - (1+\alpha) - 2\alpha(x_1^2 + x_1x_2 + x_2^2)\\
f_2\colon &\quad -2(1-\alpha)x_1 - (1-3\alpha)x_2 + (1-\alpha) - 2\alpha(x_1^2 + x_1x_2 + x_2^2)\\
f_3\colon &\quad (1+\alpha)x_1 - (1-\alpha)x_2 - 2\alpha(x_1^2 + x_1x_2 + x_2^2).
\end{aligned}$$

Note that $x_1 = x_2 = 1/3$ gives $f_1 = f_2 = f_3 = 0$, so this is our Nash equilibrium. For the replicator dynamic, we have $\dot{x}_1 + \dot{x}_2 + \dot{x}_3 = 0$, so we need only the first two equations. Assuming $x_1, x_2 > 0$, we get

$$\frac{\dot{x}_1}{x_1} = -(2\alpha(x_1^2 + x_1x_2 + x_2^2) - (1+3\alpha)x_1 - 2(1+\alpha)x_2 + (1+\alpha))$$

$$\frac{\dot{x}_2}{x_2} = -(2\alpha(x_1^2 + x_1x_2 + x_2^2) + 2(1-\alpha)x_1 + (1-3\alpha)x_2 - (1-\alpha)).$$

It is straightforward to check that $x_1 = x_2 = 1/3$ is the only fixed point for this set of equations in the positive quadrant.

The Jacobian of this system at the Nash equilibrium is

$$\frac{1}{3}\begin{bmatrix} 1+\alpha & 2 \\ -2 & -1+\alpha \end{bmatrix}.$$

This has determinant $\beta = 1/3 + \alpha^2/9 > 0$, the trace is $\mathrm{Tr} = 2\alpha/3$ and the discriminant is $\gamma = \mathrm{Tr}^2/4 - \beta = -1/3$. The eigenvalues are thus $\alpha/3 \pm \sqrt{-3}/3$, which have nonzero real parts for $\alpha \neq 0$. Therefore, the system is hyperbolic. By theorem 11.5, the dynamical system is a stable focus for $\alpha < 0$ and an unstable focus for $\alpha > 0$.

12.14 *Uta stansburiana* in Motion

It is easy to check that if the frequencies of orange-throats (rock), blue-throats (paper), and yellow-striped (scissors) are α, β, and $1-\alpha-\beta$, respectively, the payoffs to the three strategies are $1-\alpha-2\beta, 2\alpha+\beta-1$, and $\beta-\alpha$, respectively. The average payoff is zero (check this!), so the replicator dynamic equations are

$$\begin{aligned} \frac{d\alpha}{dt} &= \alpha(1-\alpha-2\beta) \\ \frac{d\beta}{dt} &= \beta(2\alpha+\beta-1). \end{aligned} \tag{A11}$$

The Jacobian matrix at the fixed point $\alpha = \beta = 1/3$ is given by

$$\begin{bmatrix} -1/3 & -2/3 \\ 2/3 & 1/3 \end{bmatrix}.$$

The trace of the Jacobian is thus zero, the determinant is $1/3 > 0$, and the discriminant is $-1/3 < 0$. By theorem 11.5 the eigenvalues are imaginary so the system is not hyperbolic. It is easy to solve for the trajectories of this system because, by theorem 11.5, they are closed orbits, and the fixed point is a center. But this tells us nothing about the original, nonlinear system (A11), because the fixed point is not hyperbolic (see theorem 11.3). So, back to the drawing board.

Let $V(\alpha,\beta,\gamma) = \ln(\alpha) + \ln(\beta) + \ln(\gamma)$. Along a trajectory of the dynamical system, we have

$$\begin{aligned} \dot{V} &= \frac{\dot{\alpha}}{\alpha} + \frac{\dot{\beta}}{\beta} + \frac{\dot{\gamma}}{\gamma} \\ &= (1-\alpha-2\beta) + (2\alpha+\beta-1) + (\beta-\alpha) = 0. \end{aligned}$$

Thus, V is constant on trajectories. This implies that trajectories are bounded and bounded away from $(0,0)$ so the set Γ of ω-limit points of a trajectory contains no fixed points, and hence by the Poincaré-Bendixson theorem (theorem 11.8), Γ is a periodic orbit. But then by theorem 11.9, Γ must contain $(0,0)$. Hence, trajectories also must spiral around $(0,0)$, and because V is increasing along a ray going northeast from the fixed point, trajectories must be closed orbits.

12.15 The Dynamics of Rock Paper Scissors

Let π_α, π_β, and π_γ be the payoffs to the three strategies. Then, we have

$$\begin{aligned}\pi_\alpha &= \beta r + (1-\alpha-\beta)s = \beta(r-s) - \alpha s + s,\\ \pi_\beta &= \alpha s + (1-\alpha-\beta)r = \alpha(s-r) - \beta r + r,\\ \pi_\gamma &= \alpha r + \beta s.\end{aligned}$$

It is easy to check that the average payoff is then

$$\begin{aligned}\bar{\pi} &= \alpha\pi_\alpha + \beta\pi_\beta + (1-\alpha-\beta)\pi_\gamma\\ &= (r+s)(\alpha+\beta-\alpha^2-\alpha\beta-\beta^2).\end{aligned}$$

At any fixed point involving all three strategies with positive probability, we must have $\pi_\alpha = \pi_\beta = \pi_\gamma$. Solving these two equations, we find $\alpha = \beta = \gamma = 1/3$, which implies that $\bar{\pi} = (r+s)/3$.

In a replicator dynamic, we have

$$\begin{aligned}\dot{\alpha} &= \alpha(\pi_\alpha - \bar{\pi}),\\ \dot{\beta} &= \beta(\pi_\beta - \bar{\pi}).\end{aligned}$$

Expanding these equations, we get

$$\begin{aligned}\dot{\alpha} &= -2\alpha\beta s - (r+2s)\alpha^2 + \alpha s + \alpha p(\alpha,\beta),\\ \dot{\beta} &= -2\alpha\beta r - (2r+s)\beta^2 + \beta r + \beta p(\alpha,\beta),\end{aligned}$$

where $p(\alpha,\beta) = (r+s)(\alpha^2+\alpha\beta+\beta^2)$.

This is, of course, a nonlinear ordinary differential equation in two unknowns. It is easy to check that its unique fixed point for $\alpha,\beta > 0$ is $\alpha = \beta = 1/3$, the mixed-strategy Nash equilibrium for this game.

For the dynamics, we linearize the pair of differential equations by evaluating the Jacobian matrix of the right-hand sides at the fixed point. The Jacobian is

$$J(\alpha,\beta) = \begin{pmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix},$$

where

$$\begin{aligned}a_{11} &= -2\beta s - 2\alpha(r+2s) + s + p(\alpha,\beta) + \alpha(2\alpha+\beta)(r+s),\\ a_{12} &= -2\alpha s + \alpha(\alpha+2\beta)(r+s),\\ a_{21} &= -2\beta r + \beta(2\alpha+\beta)(r+s),\\ a_{22} &= r - 2\alpha r - 2\beta(2r+s) + p(\alpha,\beta) + \beta(\alpha+2\beta)(r+s),\end{aligned}$$

so

$$J(1/3, 1/3) = \frac{1}{3}\begin{pmatrix} -s & r-s \\ s-r & -r \end{pmatrix}.$$

The eigenvalues of the linearized system are thus

$$\frac{1}{6}\left[-(r+s) \pm i\sqrt{3}(r-s)\right].$$

We prove the assertions as follows:

a. The determinant of the Jacobian is $(r^2 - rs + s^2)/9$. This has a minimum where $2s - r = 0$, with the value $r^2/12 > 0$. This shows that the system is hyperbolic, and because the determinant is positive, it is a node or a focus.

b. The real parts of the eigenvalues are negative if and only if $r + s > 0$ and are positive if and only if $r + s < 0$.

c. The eigenvalues are complex for $r \neq s$.

d. If $r + s = 0$, the eigenvalues are purely imaginary, so origin is a center. We thus cannot tell how the nonlinear system behaves using the linearization.

   However, we can show that the quantity $q(\alpha, \beta) = \alpha\beta(1 - \alpha - \beta)$ is constant along trajectories of the dynamical system. Assuming this (which we will prove in a moment), we argue as follows. Consider a ray R through the fixed point (1/3,1/3) pointing in the α-direction. Suppose $q(\alpha, \beta)$ is strictly decreasing along this ray (we will also prove this in a moment). Then, the trajectories of the dynamical system must be closed loops. To see this, note first that the fixed point cannot be a stable node, because if we start at a point on R near the fixed point, q decreases as we approach the fixed point, but q must be constant along trajectories, which is a contradiction. Thus, the trajectories of the system must be spirals or closed loops. But they cannot be spirals, because when they intersect R twice near the fixed point, the intersection points must be the same, because $q(\alpha, \beta)$ is constant on trajectories but decreasing on R near the fixed point.

   To see that q is decreasing along R near (1/3,1/3), note that

$$q(1/3 + t, 1/3) = \frac{1}{3}\left(\frac{1}{3} - t\right)^2,$$

which has a derivative with respect to t that evaluates to $-2/9 < 0$ at $t = 0$.

To see that $q(\alpha, \beta)$ is constant along trajectories, note that the differential equations for the dynamical system, assuming $r = 1$, $s = -1$, can be written as

$$\dot{\alpha} = 2\alpha\beta + \alpha^2 - \alpha \tag{A12}$$
$$\dot{\beta} = -2\alpha\beta - \beta^2 + \beta. \tag{A13}$$

Then,

$$\begin{aligned}\frac{d}{dt}q(\alpha,\beta) &= \frac{d}{dt}[\alpha\beta(1-\alpha-\beta)] \\ &= \beta(1-\alpha-\beta)\dot{\alpha} + \alpha(1-\alpha-\beta)\dot{\beta} + \alpha\beta(-\dot{\alpha}-\dot{\beta}) \\ &= 0,\end{aligned}$$

where we get the last step by substituting the expressions for $\dot{\alpha}$ and $\dot{\beta}$ from (AA12) and (AA13).

12.16 The Lotka-Volterra Model and Biodiversity

a. This is simple algebra, though you should check that the restrictions on the signs of a, b, c,and d ensure that $p^* > 0$.

b. We have

$$\begin{aligned}\frac{\dot{p}}{p} &= \frac{\dot{u}}{u} - \left[\frac{\dot{u}}{w} + \frac{\dot{v}}{w}\right] \\ &= \frac{\dot{u}}{u} - \left[p\frac{\dot{u}}{u} + (1-p)\frac{\dot{v}}{v}\right] \\ &= ap + b(1-p) - kw \\ &\quad - [p[ap + b(1-p) - kw] + (1-p)[cp + d(1-p) - kw]] \\ &= ap + b(1-p) - [p[ap + b(1-p)] + (1-p)[cp + d(1-p)]] \\ &= \pi_A - \bar{\pi}.\end{aligned}$$

c. We have

$$
\begin{aligned}
\frac{\dot{p}_i}{p_i} &= \frac{\dot{u}_i}{u_i} - \sum_{j=1}^{n} \frac{\dot{u}_j}{u} \\
&= \frac{\dot{u}_i}{u_i} - \sum_{j=1}^{n} \frac{\dot{u}_j}{u_j} p_j \\
&= \sum_{j=1}^{n} a_{ij} p_j - kw - \sum_{j=1}^{n} \left(\sum_{k=1}^{n} a_{jk} p_k - ku \right) p_j \\
&= \sum_{j=1}^{n} a_{ij} p_j - ku - \sum_{j,k=1}^{n} a_{jk} p_k p_j + ku \sum_{k=1}^{n} p_j \\
&= \sum_{j=1}^{n} a_{ij} p_j - \sum_{j,k=1}^{n} a_{jk} p_j p_k .
\end{aligned}
$$

This proves the assertion, and the identification of the resulting equations as a replicator dynamic is clear from the derivation.