On the General Impossibility of Persistent Unequal Exchange Free Trade Equilibria in the Pre-industrial World Economy*

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Abstract

This paper analyzes the persistency of the unequal exchange of labor (UE) in international trade. An intertemporal model of a world economy is defined with a leisure preference and no discount factor. Every incompletely specialized free trade equilibrium is characterized as having non-persistent UE, which verifies the convergence of economies without relying on economic growth or diminishing returns to scale. In particular, it characterizes a subclass of equilibria in which the sequence of real interest rates does not converge to zero, but UE tends to disappear while equivalently the distribution of capital assets tends to be equalized in the long run.

JEL classification: D51; D63; D91.

Keywords: Unequal exchange of labor; a world economy with a leisure preference; non-stationary relative prices of commodities.

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1 Introduction

Given that there are large inequalities across countries in income per capita and output per worker,[1] one of the central questions in economics is to ask what the primary mechanism is that persistently generates such disparity between rich and poor countries. The so-called convergence theory (the catch-up theory) argues that, under the assumptions of free access to production techniques and free trade, per capita incomes in poorer nations will tend to grow faster than in richer nations, due to the difference of diminishing returns. Thus, all nations would converge in terms of per capita income (more precisely, income per hours worked).

On the contrary, there have been many works which suggest that a specific institutional structure in the real world engenders the persistent feature of economic inequality between rich and poor countries. The classical literature of the dependence school, such as Emmanuel (1972), argues that institutionalized wage differentials between wealthy developed nations and poor less-developed nations result in the world economy being characterized by a systematic feature of the unequal exchange of labor between these nations.[2] Sokoloff and Engerman (2000) suggest that in the pre-modern world economy, factor endowments are a central determinant of structural inequality, but that the role of institutions is significant for the persistency of such inequality. Korotayev, Goldstone, and Zinkina (2015) discuss the crucial role of non-economic forces, such as demographics, to explain the replacement of the Great Divergence phase with the Great Convergence phase.[3]

In the theoretical works, Roemer (1982; 1983) and Yoshihara and Kaneko (2016) show that even in perfectly competitive markets, the *unequal exchange of labor* (UE) can emerge. For instance, according to the latter's work, given a simple static model of international trade, an unequal distribution of capital endowments engenders UE between rich and poor nations even as it leads to the international division of labor by which all nations can enjoy the mutual

[1]For instance, see Maddison (2001).

[2]This argument was criticized by Samuelson (1976, 1978) and Ben-David (1993). The literature related to this debate is too vast for a comprehensive list of references. For a discussion of the classic contributions, we refer the reader to the excellent reviews by Bacha (1978) and Griffin and Gurley (1985).

[3]The recent literature, for instance, Allen (2011) and Sala-i-Martin (2006), characterizes the world economy in twenty centuries as the Great Divergence until the early 1970s, which has been replaced by the Great Convergence since the late 1980s.

gains from trade. This result is also consistent with the classical view of Marx (1968, chapter 20, (e)) who notes that "a richer country exploits a poorer one, even when the latter benefits from the exchange."

As a body, these works suggest the following fundamentally theoretical question: are free trade mechanisms or non-economic factors the main source for the persistent inequality among nations? If the answer is the former, it may imply that the inequality between rich and poor nations emerges persistently in perfectly competitive environments even without any non-economic forces. If the answer is the latter, this result may give a theoretical foundation to the arguments discussed in the empirical literature, such as Sokoloff and Engerman (2000) and Korotayev et al (2015).

To address this theoretical question in this paper, we introduce a simple intertemporal model of perfectly competitive international trade. To provide a measure of inequality in our model, we apply the notion of UE. This notion is relevant to the huge literature of the dependence school and it gives us a simple measure of the income inequality per hours worked. Our subject here is then reduced to checking whether the free trade mechanism can *persistently* engender UE in perfectly competitive environments, even if the UE initially emerges due to an unequal distribution of capital among nations. The solution to this fundamental question is not obvious. The convergence theory mentioned above is not a useful guide, as it seems to rely crucially on the diminishing returns structure of macroeconomic growth models.

The answer to this question is that it is generally impossible to generate persistent UE via the free trade mechanism *alone*, even under the non-diminishing returns structure of a multi-sectoral dynamic general equilibrium model. This implies that the persistent feature of UE observed in the real world economy should be attributed to some non-economic factors outside of the perfectly competitive mechanism.

The model developed here includes, like Roemer (1982; 1983) and Yoshihara and Kaneko (2016), the simple feature that all nations can access a common Leontief production technique and have a common *leisure preference*. This preference means nations are primarily concerned with their citizens' enjoyment of free hours (or leisure time), given that a common subsistence consumption bundle, necessary for the citizens' survival, is ensured. We call such an intertemporal model a *pre-industrial world economy*,[4] since this leisure preference was ubiquitous in the pre-industrial society of early

[4] Roemer (1982) named the same economic model a *pre-chapitalist economy.*

mercantilism before the 17th century, as Kawakita (2010) argues.

With this model of a pre-industrial world economy, we develop an intertemporal general equilibrium analysis. Each nation takes present prices and an infinite sequence of expected prices and solves an optimal plan of intertemporal economic activities over an infinite horizon, subject to the standard budget and feasibility constraints. Here, each nation is assumed to have *no time preference*. The standard time preference model is not appealing for this subject, as will be discussed in more detail in section 2. Therefore, the *catching-up optimality* criterion developed by von Weizsäcker (1965) and Gale (1967) is applied to define the nation's optimization program. As in the standard model of international trade theory, all commodities can be exchanged in the world market, but neither international labor nor capital markets exist. A free trade equilibrium is specified by a profile of an infinite sequence of prices and of each nation's optimal plan of intertemporal economic activities, such that the aggregate excess demand condition is satisfied in each period of production. We call such an equilibrium a *reproducible solution* (RS).[5]

Given this equilibrium notion, we characterize almost all such equilibria in terms of the persistency of UE. The definition of UE is specified as the difference between the supply of labor hours and the sum of direct and indirect labor inputs needed to produce the subsistence commodity bundle as a net output, following the standard literature of UE exploitation theory.[6] First, the set of RSs is partitioned into two subsets: one of incompletely specialized equilibria in which factor price equalization holds at every period; and the other as the complement of the first type. Second, note that all of the RSs in the second category are characterized by *labor-value pricing* (the commodity price vector at every period is proportional to the *labor value*) and so no UE emerges. Therefore, we will focus only on the first type, which we call *incompletely specialized RSs*.

We first characterize the necessary and sufficient condition for the emergence of UE at each period in an incompletely specialized RS. UE emerges if and only if the distribution of the nations' 'real interest' incomes (nominal interest income minus speculation income obtained by utilizing the variation of commodity prices) at this period is unequal (Theorem 1 in section 3.2). This

[5] The RS discussed in this paper is an intertemporal extension of the static and stationary version of reproducible solutions discussed by Roemer (1982) and Yoshihara and Kaneko (2016).

[6] This point will be discussed in section 3.1 in detail.

theorem generalizes the so-called *Fundamental Marxian Theorem* (Okishio, 1963; Morishima, 1974; Medvegyev, 2013) of a static, stationary free trade equilibrium (Yoshihara and Kaneko, 2016) to a *non-stationary* intertemporal free trade equilibrium.

Second, we show that the observed UE in an incompletely specialized RS will disappear in the infinite limit of time (Theorem 2 in section 3.3). Thus, without any additional assumption, all nations are shown to supply the same labor hours and earn the same net income in the infinite limit of time. This may verify that economies converge through the perfectly competitive international trade mechanism alone: unlike conventional convergence theory, neither economic growth nor diminishing returns to scale is indispensable.

We also classify two types of mechanisms which drive the observed UE to vanish, according to the type of incompletely specialized RS. One type has an infinite sequence of equilibrium commodity prices which converge to a stationary price vector; and the other has no such sequence. Interestingly, in the first type of RSs, the converged stationary price vector is always the labor-value pricing, and so the 'real' interest rate converges to zero, which drives the emerging UE to disappear in the infinite limit (Corollary 3 in section 3.3). This observation resembles the standard argument of dynamic general equilibrium theory which shows the convergence of the equilibrium sequence of interest rates to the rate of time preference.

A more interesting observation relates to the second type of RS, in which equilibrium commodity prices do not converge to any stationary price vector. In this case, the associated sequence of 'real' interest rates does not converge to zero. However, the distribution of each nation's capital endowments tends to be equalized in the infinite limit even though it is unequal in the initial period, which makes UE disappear in the long run (Corollary 4 in section 3.3). This is a surprising and perhaps new observation, which is not conjectured from the standard argument of dynamic general equilibrium theory.

Before closing this section, we should explain why our special attention to the class of pre-industrial world economies is reasonable, given the fundamental question raised above. It is because our economic model with the leisure preference provides us with the simplest and coherent microfoundation for the stationary economies with no capital accumulation nor population growth. Indeed, as shown later (Proposition 4 in section 2.2), an intrinsic feature of intertemporal free trade equilibria in pre-industrial world economies is the infinite repetition of simple reproduction of a stationary capital stock with no population growth. Therefore, the main theorems obtained in such economies are completely independent of the complicated issues involved in capital accumulation and population growth.[7] It allows us to straightforwardly conclude from Corollaries 3 and 4 that, in perfectly competitive environments, the free trade mechanism *alone* cannot engender the persistent feature of UE. It also implies that the generated UE cannot persist even if scarcity of capital is preserved.

Another reason for our special attention to pre-industrial world economies is relevant to the debate over how to define the notion of UE, for which there have been many proposals (such as Okishio (1963), Morishima (1973, 1974), Roemer (1982a, chapter 5), and Dumenil (1980, 1984)-Foley (1982)).[8] Interestingly, all of these proposals are reduced to the classical definition proposed by Okishio (1963) and Morishima (1973) within the class of pre-industrial world economies,[9] and so we can focus only on the classical definition without loss of generality. Furthermore, the impossibility theorem has a strong implication that *regardless of which alternative UE-formulation is selected*, UE is unable to persist in perfectly competitive free trade environments.

The remainder of the paper is organized as follows: Section 2 presents the basic model and an equilibrium notion. Section 3.1 defines the formulation of exploitation as UE. Section 3.2 discusses the existence and characterization of free trade equilibria with and without UE, and Section 3.3 shows that the generated UE tends to disappear in the infinite long run. Finally, Section 4 concludes the paper.

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[7] Because of this feature, the main theorems in this paper are logically compatible with the analysis of Galor and Mountford (2006, 2008). First, the latter addresses the world economy in the second phase of industrial revolution, which is beyond the scope of this paper. Second, they argue that (human) capital accumulation and population growth are the main factors to generate the Great Divergence in that phase, which is completely consistent with the main results of this paper.

[8] These works suggest that, whenever a production technology more complex than the Leontief one is assumed, whether exploitative UE is observed or not in perfectly competitive economies may vary according to which alternative formulation is adopted. This issue is axiomatically analyzed by Yoshihara (2010, 2017) and Veneziani and Yoshihara (2015, 2017a, 2017b).

[9] This property is shown by Veneziani and Yoshihara (2015, 2017a) and Yoshihara (2017).

2 Model and preliminary analysis

2.1 Economy

A world economy (economy, in short) comprises a set of nations, $\mathcal{N} = \{1, .., N\}$, with generic element $\nu \in \mathcal{N}$. In an economy, there are $n \geqq 2$ types of (purely private) commodities that are transferable in markets. The production technology, commonly accessible by all nations, is a fixed Leontief production technology, (A, L), where A is an $n \times n$ non-negative matrix of input coefficients, and L is a $1 \times n$ positive vector of direct labor coefficients. Here, A is assumed to be productive and indecomposable.[10]

Assume that for each production period, the maximal amount of labor supply by every nation is equal to unity. That is, there is no difference in the size of the population or in labor skills (human capital) among nations. In every period t, each nation requires a positive vector of commodity bundle $b \in \mathbb{R}^n_{++}$, which every citizen in every nation must consume for his/her survival in one period of production; this is called the *subsistence consumption bundle*.

Let $\overline{\omega} \in \mathbb{R}^n_{++}$ be the world endowments of material input goods at the beginning of the initial period of production. Note that, given the Leontief technology (A, L), at least $\omega^{Nb} \equiv A (I - A)^{-1} (Nb)$ of material input vector must be endowed in the economy in order to reproduce the subsistence bundle Nb in every period. Therefore, we shall assume that $\overline{\omega} \geqq \omega^{Nb}$ holds.[11] Moreover, we shall also assume $\overline{\omega} \ngeqq \omega^{Nb} + Nb$, which implies a *non-free lunch* in the initial period. Likewise, at least $L (I - A)^{-1} (Nb)$ of aggregate labor must be endowed in the economy. As the maximal aggregate labor supply is given by N, we shall assume that $N > L (I - A)^{-1} (Nb)$.

As mentioned in the introduction, we shall focus on a specific type of preference profile in this economy, in that every nation has a *leisure preference* defined as follows. Let $C \equiv \left\{ c \in \mathbb{R}^n_+ \mid c \geqq b \right\} \times [0, 1]$ be the consumption space common to all nations, where $[0, 1]$ represents the space of leisure consumption. Then, every nation has the common instantaneous welfare

[10] Let J be the index set of A's dimension. Then, A is said to be *decomposable* if there is a pair of J_1 and J_2 such that $J = J_1 \cup J_2$, $J_1 \cap J_2 = \varnothing$, $J_1, J_2 \neq \varnothing$, and $a_{ij} = 0$ for $i \in J_1, j \in J_2$. If A is *indecomposable*, then: for each i, there exists j such that $a_{ij} > 0$ and $i \neq j$; and for each j, there exists i such that $a_{ij} > 0$ and $i \neq j$.

[11] For all vectors $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$, $x \geqq y$ if and only if $x_i \geqq y_i$ $(i = 1, \ldots, n)$; $x \geq y$ if and only if $x \geqq y$ and $x \neq y$; $x > y$ if and only if $x_i > y_i$ $(i = 1, \ldots, n)$.

function $u : C \to \mathbb{R}$, defined as follows: for each $(c, l) \in C$,

$$u(c, l) = 1 - l,$$

where l is the amount of labor supply. This form of common welfare function implies that no nation is concerned with an increase in consumption goods beyond the subsistence level, b, but they evaluate their social welfare in terms of the increase in free hours (leisure time), once b is ensured.

A *world economy* is thus defined by the profile $\langle \mathcal{N}, (A, L), u, \overline{\omega} \rangle$, which we call a *pre-industrial world economy* (*economy*, in short). This economy is assumed to have the time structure of production, such that production takes one period. The material inputs available in the present period cannot exceed the amount of material input commodities accumulated by the end of the preceding period of production.

In such an economy, each nation decides its plan of economic activities over an infinite-horizon of periods. Unlike the standard macroeconomic intertemporal models, we assume that every nation has no time preference and so no discount factor is introduced. Let us explain why we focus on the case of no time preference. First, if a time preference is introduced and the optimization problem is given by the maximization of the infinite additive sum of discounted one-period welfare functions, then UE emerges persistently. This is shown because the equilibrium sequence of interest rates converges to a positive rate of time preference in this setting. Second, the general normative significance of time preference has been questioned by many economists and political philosophers (include Ramsey, 1928; Rawls, 1971; Sidgwick,1907). While time discounting may be relevant in the positive analysis of individual choices, as Veneziani (2013) emphatically argues, the role of pure time preference in exploitation theories is much less obvious, since an explanation of persistent UE based on time preference is not appealing from the normative perspective of exploitative UE.

To deal with the case of non-discounting, we will employ the concept of *catching-up optimality* developed by von Weizsäcker (1965) and Gale (1967), which is formulated as follows:

For $t = 1, 2, \ldots$, let p_{t-1} be the $1 \times n$ vector of the market prices of commodities at the end of period $t - 1$, which is identical to the beginning of period t, and let (w_t^ν, r_t^ν) be the wage rate and the nominal interest rate in ν's domestic market at the end of period t. Moreover, let $(\omega_1^\nu)_{\nu \in \mathcal{N}} \in \mathbb{R}_+^{nN}$ be a profile of initial endowments of material input goods satisfying $\sum_{\nu \in \mathcal{N}} \omega_1^\nu = \overline{\omega}$.

Given a price sequence, $\left\{p_{t-1};\left(w_t^\nu,r_t^\nu\right)_{\nu\in\mathcal{N}}\right\}_{t=1}^{\infty}$, each nation ν chooses a path $\left\{\left(x_t^\nu,\delta_t^\nu,\omega_{t+1}^\nu\right)\right\}_{t=1}^{\infty}$ such that: for each period, (1) wealth carried over from the previous period, $p_{t-1}\omega_t^\nu$, is sufficient to purchase the material inputs, Ax_t^ν, for the production activity, x_t^ν, and the commodities for speculative purposes, δ_t^ν, at the beginning of the present period; (2) labor inputs $l_t^\nu = Lx_t^\nu$ do not exceed the amount of available labor; (3) earnings at the end of the present period, $p_t x_t^\nu + p_t\delta_t^\nu$, are sufficient for consuming b and carrying over $p_t\omega_{t+1}^\nu$ to the next period; (4) total net revenue from the production, $p_t x_t^\nu - p_{t-1}Ax_t^\nu$, is divided into total wage income, $w_t^\nu Lx_t^\nu$, and total interest income, $r_t^\nu p_{t-1}Ax_t^\nu$, through the domestic market. Formally, for $t = 1, 2, \ldots$,

$$
\begin{aligned}
& p_{t-1}Ax_t^\nu + p_{t-1}\delta_t^\nu \leqq p_{t-1}\omega_t^\nu;\ (1)\\
& l_t^\nu = Lx_t^\nu \leqq 1;\ (2)\\
& p_t x_t^\nu + p_t\delta_t^\nu \geqq p_t b + p_t\omega_{t+1}^\nu;\ (3)\\
& p_t x_t^\nu - p_{t-1}Ax_t^\nu = w_t^\nu Lx_t^\nu + r_t^\nu p_{t-1}Ax_t^\nu;\ (4)\\
& \omega_1^\nu \in \mathbb{R}_+^n \text{ given}; \quad x_t^\nu, \delta_t^\nu, \omega_{t+1}^\nu \in \mathbb{R}_+^n.
\end{aligned}
$$

For convenience, we use the following definitions and notation. Given a price sequence $\left\{p_{t-1};\left(w_t^\nu,r_t^\nu\right)_{\nu\in\mathcal{N}}\right\}_{t=1}^{\infty}$, a set of all *possible production activity paths (from ω_1^ν)* is defined by

$$
\mathbf{x}\left(\omega_1^\nu\right) \equiv \left\{\left(x_1^\nu, x_2^\nu, \ldots\right) \mid \left(x_t^\nu,\delta_t^\nu,\omega_{t+1}^\nu\right) \in \mathbb{R}_+^{3n} \text{ satisfies (1)-(4) for all } t = 1,2,\ldots\right\}.
$$

A possible production activity path $\{x_t^\nu\}_{t=1}^{\infty} \in \mathbf{x}\left(\omega_1^\nu\right)$ is *catching-up optimal (from ω_1^ν)* if any possible production activity path $\{x_t'\}_{t=1}^{\infty} \in \mathbf{x}\left(\omega_1^\nu\right)$ satisfies the following:

$$
\liminf_{T\to\infty} \sum_{t=1}^{T} \left(Lx_t' - Lx_t^\nu\right) \geqq 0.
$$

Before defining the concept of equilibrium, we show a necessary condition for catching-up optimal paths:

Proposition 1: Given $\left\{p_{t-1};\left(w_t^\nu,r_t^\nu\right)_{\nu\in\mathcal{N}}\right\}_{t=1}^{\infty}$ with $p_{t-1} > \mathbf{0}$ and $w_t^\nu > 0$ $(\forall\nu\in\mathcal{N})$ for all $t = 1,2,\ldots$, let $\{x_t^\nu\}_{t=1}^{\infty} \in \mathbf{x}\left(\omega_1^\nu\right)$, with $x_t^\nu \geq \mathbf{0}$ $(t = 1,2,\ldots)$, be catching-up optimal for every $\nu\in\mathcal{N}$. Then,

$$
w_{t+1}^\nu = \left(1 + r_{t+1}^\nu\right) w_t^\nu \qquad (5)
$$

holds for all $t = 1,2,\ldots$.

Proof. See Appendix. □

2.2 Equilibrium and its Preliminary Results

We now introduce the concept of equilibrium for our model.

Definition 1: For an economy $\langle \mathcal{N}, (A, L), u, (\omega_1^\nu)_{\nu \in \mathcal{N}} \rangle$, a *reproducible solution* (RS) is a price system $\left\{p_{t-1}^*; (w_t^{\nu*}, r_t^{\nu*})_{\nu \in \mathcal{N}}\right\}_{t=1}^\infty$ and an associated possible allocation $\left\{\left(x_t^{*\nu}, \delta_t^{*\nu}, \omega_{t+1}^{*\nu}\right)_{\nu \in \mathcal{N}}\right\}_{t=1}^\infty$ such that:

(a) $\{x_t^{*\nu}\}_{t=1}^\infty \in \mathbf{x}(\omega_1^\nu)$ is catching-up optimal for every $\nu \in \mathcal{N}$;

(b) $x_t^* + \delta_t^* \geqq Nb + \omega_{t+1}^*$ for all $t = 1, 2, \ldots$; and

(c) $Ax_t^* + \delta_t^* \leqq \omega_t^*$ for all $t = 1, 2, \ldots$,

where $x_t^* \equiv \sum_{\nu \in \mathcal{N}} x_t^{*\nu}$, $\delta_t^* \equiv \sum_{\nu \in \mathcal{N}} \delta_t^{*\nu}$, and $\omega_{t+1}^* \equiv \sum_{\nu \in \mathcal{N}} \omega_{t+1}^{*\nu}$.

Definition 1 states that in an RS, taking the price system $\left\{p_{t-1}^*; (w_t^{\nu*}, r_t^{\nu*})_{\nu \in \mathcal{N}}\right\}_{t=1}^\infty$, (a) every nation chooses its own optimal plan; (b) the standard excess demand condition is satisfied; and finally, (c) the aggregate activities of production are feasible under the stock of material input goods at each period.

In what follows, we will devote special attention to the subset of equilibria with *incomplete specialization*, in which each nation produces all commodities.

Definition 2: For an economy $\langle \mathcal{N}, (A, L), u, (\omega_1^\nu)_{\nu \in \mathcal{N}} \rangle$, an RS, $\left\langle \left\{p_{t-1}^*; (w_t^{\nu*}, r_t^{\nu*})_{\nu \in \mathcal{N}}\right\}_{t=1}^\infty, \left\{\left(x_t^{*\nu}, \delta_t^{*\nu}, \omega_{t+1}^{*\nu}\right)_{\nu \in \mathcal{N}}\right\}_{t=1}^\infty \right\rangle$, is said to be *incompletely specialized* if it satisfies $x_t^{*\nu} > \mathbf{0}$ for every $\nu \in \mathcal{N}$ at every $t = 1, 2, \ldots$.

Note that if an RS is incompletely specialized, then

$$p_t^* = (1 + r_t^{\nu*})\, p_{t-1}^* A + w_t^{\nu*} L > \mathbf{0} \qquad (6)$$

holds for every $\nu \in \mathcal{N}$ by $x_t^{*\nu} > \mathbf{0}$ at every t, as $p_t^* \leqq (1 + r_t^{\nu*})\, p_{t-1}^* A + w_t^{\nu*} L$ is a necessary condition of an equilibrium price vector.

It is possible to develop some preliminary results toward the characterization of incompletely specialized RSs. First, factor price equalization among nations can be established under incompletely specialized RSs.

Proposition 2 [Factor Price Equalization]**:** Given an economy $\langle \mathcal{N}, (A, L), u, (\omega_1^\nu)_{\nu \in \mathcal{N}} \rangle$, let $\left\{p_{t-1}^*; (w_t^{\nu*}, r_t^{\nu*})_{\nu \in \mathcal{N}}\right\}_{t=1}^\infty$ be an incompletely specialized RS price system. Then, for each period t, the followings hold:

(i) if there exists at least one pair of i and j such that $i \neq j$ and $\frac{\left(p_{t-1}^{*}A\right)_i}{L_i} \neq \frac{\left(p_{t-1}^{*}A\right)_j}{L_j}$, then $\left(w_t^{\nu *}, r_t^{\nu *}\right) = \left(w_t^{\mu *}, r_t^{\mu *}\right)$ for every $\nu, \mu \in \mathcal{N}$;
(ii) if $\left(w_t^{\nu *}, r_t^{\nu *}\right) = \left(w_t^{\mu *}, r_t^{\mu *}\right)$ for every $\nu, \mu \in \mathcal{N}$, then $\left(w_{t+1}^{\nu *}, r_{t+1}^{\nu *}\right) = \left(w_{t+1}^{\mu *}, r_{t+1}^{\mu *}\right)$ for every $\nu, \mu \in \mathcal{N}$.

Proof. See Appendix. □

By Proposition 2, *factor prices are equalized across all nations for every period* under any incompletely specialized RS whenever $p_0^* A$ and L are linearly independent. Except in a rare case, it is always possible to select such a p_0^*. Therefore, in the following discussion we may focus on those incompletely specialized RSs in which factor prices are equalized among nations for every period, without loss of generality.

As we argued above, the price system of any incompletely specialized RS must satisfy conditions (5) and (6). Therefore, without loss of generality, we can specify the class of equilibrium prices of incompletely specialized RSs. Formally, let us denote a sequence of interest rates $\{r_t\}_{t=1}^{\infty} \subset \overline{\mathbb{R}}_+$ by $\mathbf{r}$, where $\overline{\mathbb{R}} \equiv \mathbb{R} \cup \{\infty\}$ is the extended real line. Let $p_0 > \mathbf{0}$ be such that $p_0 A$ and L are linearly independent. Then, given such $p_0 > \mathbf{0}$, and $w_0 > 0$ and $\mathbf{r}$, let us denote the price sequence $\{p_{t-1}; (w_t, r_t)\}_{t=1}^{\infty}$ with $w_t = (1 + r_t) w_{t-1}$ and $p_t = (1 + r_t) p_{t-1} A + w_t L$ for each $t = 1, 2, \ldots$, by $\mathbf{p}(p_0, w_0, \mathbf{r})$. Therefore, when we write $\{p_{t-1}; (w_t, r_t)\}_{t=1}^{\infty} = \mathbf{p}(p_0, w_0, \mathbf{r})$, this implies the sequence $\{p_{t-1}; (w_t, r_t)\}_{t=1}^{\infty}$ satisfies conditions (5) and (6); $\{r_t\}_{t=1}^{\infty} = \mathbf{r}$; and its commodity price and wage rate at $t = 0$ are p_0 and w_0. Thus, if $\{p_{t-1}; (w_t, r_t)\}_{t=1}^{\infty}$ is the equilibrium price system of an incompletely specialized RS, then $\{p_{t-1}; (w_t, r_t)\}_{t=1}^{\infty} = \mathbf{p}(p_0, w_0, \mathbf{r})$ must hold.

Let $v \equiv L (I - A)^{-1}$ be called the *labor value vector*. By the assumptions of A and L, $v > \mathbf{0}$ holds. Now, we can provide the second preliminary result toward the characterizations of incompletely specialized RSs:

Proposition 3: Let $\{p_{t-1}; (w_t, r_t)\}_{t=1}^{\infty} = \mathbf{p}(p_0, w_0, \mathbf{r})$ for $\mathbf{r} \subset \overline{\mathbb{R}}_+$. If $(p_0, w_0) \in \mathbb{R}_+^n \times \mathbb{R}_{++}$ satisfies $p_0 \geqq p_0 A + w_0 L > \mathbf{0}$, then $(1 + r_t) p_{t-1} - p_t \geqq \mathbf{0}$ holds for all $t = 1, 2, \ldots$. Specifically, the following hold:

(i) $(1 + r_t) p_{t-1} - p_t \geq \mathbf{0}$ for all $t = 1, 2, \ldots$ if and only if $p_0 \geq p_0 A + w_0 L$;

(ii) There exists $q > \mathbf{0}$ such that for each $t = 0, 1, \ldots$, $p_t = \varsigma_t q$ holds for some $\varsigma_t > 0$ if and only if there exists $\varsigma_v > 0$ such that $q = \varsigma_v v$.

Proof. See Appendix. □

Proposition 3(ii) implies that an equilibrium price system in an incompletely specialized RS is characterized by stationary relative prices of commodities if and only if they are *labor-value pricing*, i.e., the price vectors are proportional to the labor-value vector through time. Such a case is derived in a quite limited situation as the following corollary suggests.

Corollary 1: For $\mathbf{r} \subset \overline{\mathbb{R}}_+$, let $\{p_{t-1}; (w_t, r_t)\}_{t=1}^{\infty} = \mathbf{p}(p_0, w_0, \mathbf{r})$ with $p_0 \geqq p_0 A + w_0 L$. Then, the following two are equivalent:

(i) for each $t = 0, 1, \ldots$, there exists $\varsigma_t > 0$ such that $p_t = \varsigma_t v$;

(ii) p_0 and L are the Perron-Frobenius eigenvector for A or $p_0 = p_0 A + w_0 L$.

In summary, Proposition 3(ii) and Corollary 1 imply that an *equilibrium price system in an incompletely specialized RS is characterized by stationary relative prices of commodities* if and only if either *the initial price vector of commodities is labor-value pricing* or *this vector and the labor coefficient vector are the Perron-Frobenius eigenvector.*[12]

Therefore, unless the condition of Corollary 1(ii) is satisfied, the equilibrium price system of the incompletely specialized RS is characterized as a non-stationary relative price sequence of commodities. Note that such an equilibrium price system $\mathbf{p}(p_0, w_0, \mathbf{r})$ may have the feature that $r_t > R$ ($\forall t = 1, 2, \ldots$), where $R > 0$ is such that $\frac{1}{1+R}$ is the Perron-Frobenius eigenvalue of A,[13] which will be discussed in Section 3.3 below.

[12] Note that, from Proposition 3 and Corollary 1, the focus of our attention on the class of incompletely specialized RSs is relevant to the main purpose of this paper. Indeed, when we consider $p_0^* A$ and L to be linearly dependent in order to allow the case of RSs with unequal factor prices among nations, it implies that p_0^* is labor-value pricing according to Yoshihara and Kaneko (2016; Lemma 1). Then, by Corollary 1, p_t^* is labor-value pricing for every $t = 1, 2, \ldots$. Then, as argued later in section 3.2, such an RS does not involve any exploitation from the initial period, which is outside of our concerns in this paper.

[13] This is in contrast with the general observation of the stationary equilibrium prices. Indeed, with a Leontief production technology (A, L), when a stationary equilibrium price system is given by $p = (1 + r) pA + wL$ for some $r \geqq 0$ and $w > 0$, it follows that $r \in [0, R)$.

Given an incompletely specialized RS, the associated sequence of feasible allocations $\left\{\left(x_t^{*\nu}, \delta_t^{*\nu}, \omega_{t+1}^{*\nu}\right)_{\nu \in \mathcal{N}}\right\}_{t=1}^{\infty}$ is called an *RS-allocation*. Let us investigate RS-allocations in the rest of this section.

To simplify the argument, we will focus on the case $\overline{\omega} = \omega^{Nb}$. The assumption, $\overline{\omega} = \omega^{Nb}$, can be justified by the fact that it corresponds to the long-run state of capital accumulation.[14] Then, it can be shown that the aggregate production activity x_t^* in any period t of an RS is identical to $(I-A)^{-1}(Nb)$, while the aggregate stock of material inputs for the next period's production, ω_{t+1}^*, is equal to ω^{Nb}.

Proposition 4: Let an economy $\left\langle \mathcal{N}, (A, L), u, (\omega_1^\nu)_{\nu \in \mathcal{N}} \right\rangle$ satisfy $\overline{\omega} = \omega^{Nb}$. Then, any RS allocation $\left\{\left(x_t^{*\nu}, \delta_t^{*\nu}, \omega_{t+1}^{*\nu}\right)_{\nu \in \mathcal{N}}\right\}_{t=1}^{\infty}$ satisfies $x_t^* = (I-A)^{-1}(Nb)$, $\delta_t^* = \mathbf{0}$, and $\omega_{t+1}^* = \omega^{Nb}$ for all $t = 1, 2, \ldots$.

Proof. See Appendix. □

By Proposition 4, the economy as a whole does not have capital accumulation. Moreover, under $\overline{\omega} = \omega^{Nb}$, any RS involves $\delta_t^* = \mathbf{0}$ $(t = 1, 2, \ldots)$.

In the following discussion of this subsection, we will show that under $\overline{\omega} = \omega^{Nb}$, only allocations with $\delta_t^{*\nu} = \mathbf{0}$ for any $\nu \in \mathcal{N}$ $(t = 1, 2, \ldots)$ are consistent with an RS. First, the following lemma establishes that if a catching-up optimal path is truncated after T periods, it solves the corresponding finite-time program with the terminal time T.

Lemma 1: Let $\{p_{t-1}; (w_t, r_t)\}_{t=1}^{\infty} = \mathbf{p}(p_0, w_0, \mathbf{r})$ and a path $\left\{\left(x_t^\nu, \delta_t^\nu, \omega_{t+1}^\nu\right)\right\}_{t=1}^{\infty}$ be catching-up optimal from $\omega_1^\nu \in \mathbb{R}_+^n \setminus \{\mathbf{0}\}$. Arbitrarily fix any number T with the corresponding stock ω_{T+1}^ν. Then, the path $\left\{\left(x_t^\nu, \delta_t^\nu, \omega_{t+1}^\nu\right)\right\}_{t=1}^{T}$ from $t = 1$ until $t = T$ solves the following program:

$$(MP_T^\nu) \qquad \min_{\{(x_t, \delta_t, \omega_{t+1})\}_{t=1}^T} \sum_{t=1}^{T} L x_t$$

subject to: for each $t = 1, 2, \ldots, T$, the constraints (1)-(3) hold;

$$\omega_1 = \omega_1^\nu; \text{ and } p_T \omega_{T+1} = p_T \omega_{T+1}^\nu.$$

[14]In fact, it can be proved that, even if $\overline{\omega} \geq \omega^{Nb}$, any RS allocation involves $\omega_t^* \to \omega^{Nb}$ as $t \to \infty$. For a more detailed argument, see Lemma II, Propositions A.2 and A.3 in the Addendum of this paper.

Proof. See Appendix. □

Secondly, the next lemma shows that the form of $\{(x_t^\nu, \mathbf{0}, Ax_t^\nu)\}_{t=1}^\infty$ can constitute a catching-up optimal solution by examining the corresponding myopic program (MP_T^ν).

Lemma 2: Taking $(p_0, w_0) \in \mathbb{R}_+^n \times \mathbb{R}_{++}$ and $\mathbf{r} \subset \overline{\mathbb{R}}_+$ such that $p_0 \geqq p_0 A + w_0 L$, let $\{p_{t-1}; (w_t, r_t)\}_{t=1}^\infty = \mathbf{p}(p_0, w_0, \mathbf{r})$. Given $\omega_1^\nu, \omega_{T+1}^\nu \in \mathbb{R}_+^n \backslash \{\mathbf{0}\}$, let $\{(x_t^\nu, \mathbf{0}, Ax_t^\nu)\}_{t=1}^T \subset \mathbb{R}_+^{3n}$ fulfill $p_{t-1}Ax_t^\nu = p_{t-1}\omega_t^\nu$, $Lx_t^\nu \leqq 1$, and $p_t (I - A) x_t^\nu = p_t b$ for $t = 1, 2, \ldots, T$. Then, it constitutes a solution to (MP_T^ν).

Proof. See Appendix. □

By Proposition 4 and Lemmas 1 and 2, we can summarize.

Corollary 2: Let an economy $\left\langle \mathcal{N}, (A, L), u, (\omega_1^\nu)_{\nu \in \mathcal{N}} \right\rangle$ satisfy $\overline{\omega} = \omega^{Nb}$. Taking $(p_0, w_0) \in \mathbb{R}_+^n \times \mathbb{R}_{++}$ and $\mathbf{r} \subset \overline{\mathbb{R}}_+$ such that $p_0 \geqq p_0 A + w_0 L$, let $\{p_{t-1}; (w_t, r_t)\}_{t=1}^\infty = \mathbf{p}(p_0, w_0, \mathbf{r})$. Then, a feasible plan $\{(x_t^\nu, \mathbf{0}, Ax_t^\nu)\}_{t=1}^\infty$ which satisfies all of the constraints (1)-(3) and fulfills $p_{t-1}Ax_t^\nu = p_{t-1}\omega_t^\nu$, $Lx_t^\nu \leqq 1$, and $p_t (I - A) x_t^\nu = p_t b$ for $t = 1, 2, \ldots, T$, constitutes a catching-up optimal solution.

Thus, we hereafter omit $\delta_t^{*\nu}$ and denote an RS allocation by $\left\{ \left(x_t^{*\nu}, \omega_{t+1}^{*\nu} \right)_{\nu \in \mathcal{N}} \right\}_{t=1}^\infty$. Moreover, under $\overline{\omega} = \omega^{Nb}$ where no RS involves capital accumulation (by Proposition 4), we may focus only on the types of RSs where no agent saves. That is, $p_t \omega_{t+1}^{*\nu} = p_t A x_t^{*\nu}$ holds for any $\nu \in \mathcal{N}$ and every t, which we call *interior RS* (IRS) following Veneziani (2007).

3 Characterizations of free trade equilibrium in terms of UE

3.1 Exploitation as UE

In this subsection, we introduce a definition of exploitation as UE. Note that while the notion of exploitation primarily refers to an asymmetric *relationship* between agents, there have been many discussions about what the exploitation is. The notion of exploitation as UE has been central in exploitation theories as argued in the recent literature such as Wright (2000) and Vrousalis (2013). There are some alternative proposals on exploitation of factors other than labor, such as Roemer (1982a).[15] However, even within the UE theory of exploitation, there have been debates over how to formulate the notion of exploitation as UE, and indeed many alternative proposals exist, such as Okishio (1963), Morishima (1973, 1974), Roemer (1982a, chapter 5), and Dumenil (1980, 1984)-Foley (1982), etc.

Given this debate, the recent literature, such as Yoshihara (2010, 2017), Yoshihara and Veneziani (2009), and Veneziani and Yoshihara (2015, 2017a, 2017b, 2018), develops an axiomatic study on reasonable formal definition of exploitation. As Yoshihara (2017) and Veneziani and Yoshihara (2015, 2017a) argue, whatever the outcome of these debates, any formal definition of UE-exploitation within the admissible domain can be reduced to the following *uniform* definition within the class of economies with the leisure preference and Leontief production technology, which was originated by Roemer (1982, Definitions 2.2)[16]:

Definition 5: For any economy $\left\langle \mathcal{N}, (A, L), u, (\omega_1^\nu)_{\nu \in \mathcal{N}} \right\rangle$, let $\left\{ p_{t-1}^*; (w_t^*, r_t^*) \right\}_{t=1}^{\infty}$ be an imperfectly specialized RS at period t associated with $\left\{ \left(x_t^{*\nu}, \delta_t^{*\nu}, \omega_{t+1}^{*\nu} \right)_{\nu \in \mathcal{N}} \right\}_{t=1}^{\infty}$. Then, the amount of socially necessary labor required to produce b as a net

[15] Fleurbaey (2014) characterizes the alternative theories of exploitation including the UE theory and Roemer's (1982a) theory of property-relational exploitation.

[16] The UE formulated in Definition 5 is worth calling exploitation, as Roemer (1982, p. 168) emphasized as follows: for some producers μ and ν, $Lx_t^\mu > vb > Lx_t^\nu$ at period t. That is, "Producer ν is working less time than is socially necessary to reproduce himself and producer μ is working longer than is socially necessary. Producer ν is exploiting μ. This comes about because .... ν is wealthier than μ, and is able to use his wealth as leverage through the exchange mechanism to force μ to work "for" him. .... That this phenomenon deserves to be called Marxian exploitation can be seen by the following. Suppose μ and ν were the only producers in the economy, and ν expropriated μ's endowment and killed him. Producer ν would now be wealthier than before; yet, at any RS for the new economy in which only he is a member, he will have to work time vb, longer than when μ was there. Thus, exploitation is an explicitly social phenomenon: ν can get away with working less than vb only because there is someone else working more than vb, to "support" him. Producer ν appears to be gaining at the expense of μ, ... even though all producers gain from trade at a reproducible solution."

output is

$$\frac{1}{N}L\left(\sum_{\nu\in\mathcal{N}}x_t^{*\nu}\right)=L\left(I-A\right)^{-1}b=vb.$$

Moreover, for each nation $\nu\in\mathcal{N}$, the supply of labor hours to earn revenue p_t^*b for its own survival is $Lx_t^{*\nu}$ at period t, and then

$$\nu \text{ is an } \textit{exploiting} \text{ nation at period } t \Longleftrightarrow Lx_t^{*\nu}<vb;$$
$$\nu \text{ is an } \textit{exploited} \text{ nation at period } t \Longleftrightarrow Lx_t^{*\nu}>vb.$$

Denoting the sets of exploiters and exploited nations at period t by $\mathcal{N}_t^{ter}$ and $\mathcal{N}_t^{ted}$, respectively, we also use the following definition, which was again originated by Roemer (1982, Definitions 2.2).

Definition 6: For any economy, $\left\langle\mathcal{N},(A,L),u,(\omega_1^\nu)_{\nu\in\mathcal{N}}\right\rangle$, let $\left\{p_{t-1}^*;(w_t^*,r_t^*)\right\}_{t=1}^{\infty}$ be an imperfectly specialized RS associated with $\left\{\left(x_t^{*\nu},\delta_t^{*\nu},\omega_{t+1}^{*\nu}\right)_{\nu\in\mathcal{N}}\right\}_{t=1}^{\infty}$. Then, $\left(x_t^{*\nu},\delta_t^{*\nu},\omega_{t+1}^{*\nu}\right)_{\nu\in\mathcal{N}}$ is *inegalitarian* at period t if and only if $\mathcal{N}_t^{ter}\neq\varnothing$ and $\mathcal{N}_t^{ted}\neq\varnothing$.

Thus, an inegalitarian RS allocation at period t implies the generation of exploitative relations at this period. By contrast, we can state that for any economy $\left\langle\mathcal{N},(A,L),u,(\omega_1^\nu)_{\nu\in\mathcal{N}}\right\rangle$, an RS allocation at period t, $\left(x_t^{*\nu},\delta_t^{*\nu},\omega_{t+1}^{*\nu}\right)_{\nu\in\mathcal{N}}$, is *egalitarian* if and only if $Lx_t^{*\nu}=Lx_t^{*\mu}$ for all $\nu,\mu\in\mathcal{N}$.

The form of exploitation in Definition 5 is applied to the context of international economies by Roemer (1982) and Yoshihara and Kaneko (2016). As mentioned above, when a more general class of production technology is considered, many alternative definitions of exploitation have been proposed. However, all of these alternatives can be regarded as an extension of Definition 5 into the class of such general economies. Therefore, the following characterization of the persistency of inegalitarian RSs is free from debate over the proper definition of labor exploitation. These results can be applied independently of this particular definition of exploitation.

3.2 Characterizations

This subsection provides a characterization for the equilibria involving UE, which is based on the exploitative relationship. Given an incompletely specialized IRS, where its associated RS allocation can have the form of $\left\{\left(x_t^{*\nu}, Ax_t^{*\nu}\right)_{\nu\in\mathcal{N}}\right\}_{t=0}^{\infty}$ by Corollary 2, we characterize the necessary and sufficient condition for this RS to involve UE. As a preliminary step, let $\omega^b \equiv \frac{\omega^{Nb}}{N}$.

Theorem 1: Given an economy $\left\langle \mathcal{N}, (A, L), u, (\omega_1^\nu)_{\nu\in\mathcal{N}} \right\rangle$ with $\overline{\omega} = \omega^{Nb}$, let $\left\{p_{t-1}^*; (w_t^*, r_t^*)\right\}_{t=1}^{\infty} = \mathbf{p}\left(p_0^*, w_0^*, \mathbf{r}^*\right)$ be an IRS with $\left\{\left(x_t^{*\nu}, Ax_t^{*\nu}\right)_{\nu\in\mathcal{N}}\right\}_{t=1}^{\infty}$. Then, for each $t = 1, 2, \ldots$, this RS is inegalitarian at period t if and only if there exists $\nu \in \mathcal{N}$ such that $\left(1+r_t^*\right)p_{t-1}^*\omega_t^{*\nu} - p_t^*\omega_{t+1}^{*\nu} \neq \left(1+r_t^*\right)p_{t-1}^*\omega^b - p_t^*\omega^b$.

Proof. For each $t = 1, 2, \ldots$, let $\omega_{t+1}^{*\nu} = Ax_t^{*\nu}$ ($\nu \in \mathcal{N}$). By Proposition 4, we can see that $\omega_t = \sum_{\nu\in\mathcal{N}} \omega_1^\nu = \omega^{Nb}$ for all $t = 1, 2, \ldots$. First, consider the case of $\left(1+r_t^*\right)p_{t-1}^* = p_t^*$ for all $t = 1, 2, \ldots$, which is equivalent to $p_0^* = p_0^*A + w_0^*L$ by (i) of Proposition 3. Note that constraint (1) yields $p_{t-1}^*\omega_t^{*\nu} = p_{t-1}^*\omega_{t+1}^{*\nu}$ ($\nu \in \mathcal{N}$) for all $t = 1, 2, \ldots$. Fix $t = 1, 2, \ldots$ arbitrarily. Then, $\left(1+r_t^*\right)p_{t-1}^* = p_t^*$ and $p_{t-1}^*\omega_t^{*\nu} = p_{t-1}^*\omega_{t+1}^{*\nu}$ ($\nu \in \mathcal{N}$) imply

$$\left(1+r_t^*\right)p_{t-1}^*\omega_t^{*\nu} - p_t^*\omega_{t+1}^{*\nu} = 0 = \left(1+r_t^*\right)p_{t-1}^*\omega^b - p_t^*\omega^b$$

for any $\nu \in \mathcal{N}$. On the other hand, from constraints (3) with equality, (4), and $\left(1+r_t^*\right)p_{t-1}^* = p_t^*$, we derive

$$w_t^*Lx_t^{*\nu} = p_t^*\left(I-A\right)x_t^{*\nu} = p_t^*b$$

for any $\nu \in \mathcal{N}$. Moreover, since $x_t^* = \left(I-A\right)^{-1}\left(Nb\right)$ as proven by Proposition 4,

$$p_t^*b = p_t^*\left(I-A\right)\frac{x_t^*}{N} = w_t^*L\frac{x_t^*}{N} = w_t^*vb.$$

Altogether, under the case of $\left(1+r_t^*\right)p_{t-1}^* = p_t^*$ for all $t = 1, 2, \ldots$ considered here, $\left(1+r_t^*\right)p_{t-1}^*\omega_t^{*\nu} - p_t^*\omega_{t+1}^{*\nu} = \left(1+r_t^*\right)p_{t-1}^*\omega^b - p_t^*\omega^b$ and $Lx_t^{*\nu} = vb$ must hold for any $\nu \in \mathcal{N}$.

Next, we consider the case of $\left(1+r_t^*\right)p_{t-1}^* \geq p_t^*$ for some t, which follows from $p_0^* \geq p_0^*A + w_0^*L$ by (i) of Proposition 3. Then, the constraints (3) with equality and (4) imply that, for each $t = 1, 2, \ldots$,

$$p_t^*b + p_t^*\omega_{t+1}^{*\nu} = p_t^*x_t^{*\nu} = \left(1+r_t^*\right)p_{t-1}^*\omega_{t+1}^{*\nu} + w_t^*Lx_t^{*\nu} \ (\forall \nu \in \mathcal{N});$$
$$Np_t^*b + p_t^*\omega_{t+1} = p_t^*x_t^* = \left(1+r_t^*\right)p_{t-1}^*\omega_{t+1} + w_t^*Lx_t^*,$$

and the constraint (1) yields that for each $t = 1, 2, \ldots$, $p_{t-1}^{*}\omega_{t}^{*\nu} = p_{t-1}^{*}\omega_{t+1}^{*\nu}$ ($\nu \in \mathcal{N}$). Since $(1+r_t^*)\, p_{t-1}^{*} - p_t^* \geq \mathbf{0}$ and $p_{t-1}^{*}\omega_{t}^{*\nu} = p_{t-1}^{*}\omega_{t+1}^{*\nu}$, we have $(1+r_t^*)\, p_{t-1}^{*}\omega_{t}^{*\nu} - p_t^*\omega_{t+1}^{*\nu} > 0$, as well as $(1+r_t^*)\, p_{t-1}^{*}\omega_t - p_t^*\omega_{t+1} > 0$ given that $\omega_t = \omega_{t+1} = \omega^{Nb}$. Let $\pi_t^\nu \equiv \frac{(1+r_t^*)p_{t-1}^*\omega_t^{*\nu}}{p_t^*\omega_{t+1}^{*\nu}} - 1 > 0$ and $\pi_t^* \equiv \frac{(1+r_t^*)p_{t-1}^*\omega_t}{p_t^*\omega_{t+1}} - 1 > 0$. Note that $(1+r_t^*)\, p_{t-1}^{*} - p_t^* \geq \mathbf{0}$ is a necessary and sufficient condition for $\pi_t^\nu = \frac{(1+r_t^*)p_{t-1}^*\omega_t^{*\nu}}{p_t^*\omega_{t+1}^{*\nu}} - 1 > 0$ and $\pi_t^* = \frac{(1+r_t^*)p_{t-1}^*\omega_t}{p_t^*\omega_{t+1}} - 1 > 0$. Then, for each $t = 1, 2, \ldots$,

$$\begin{aligned}
p_t^* b + p_t^*\omega_{t+1}^{*\nu} &= (1+r_t^*)\, p_{t-1}^*\omega_t^{*\nu} + w_t^* L x_t^{*\nu} = (1+r_t^*)\, p_{t-1}^*\omega_{t+1}^{*\nu} + w_t^* L x_t^{*\nu} \\
&= (1+\pi_t^\nu)\, p_t^*\omega_{t+1}^{*\nu} + w_t^* L x_t^{*\nu} \;\; (\forall \nu \in \mathcal{N}); \\
N p_t^* b + p_t^*\omega_{t+1} &= (1+r_t^*)\, p_{t-1}^*\omega_t + w_t^* L x_t^{*} = (1+r_t^*)\, p_{t-1}^*\omega_{t+1} + w_t^* L x_t^{*} \\
&= (1+\pi_t^*)\, p_t^*\omega_{t+1} + w_t^* L x_t^{*}.
\end{aligned}$$

These yield, for each $t = 1, 2, \ldots$, $p_t^* b = \pi_t^\nu p_t^* \omega_{t+1}^{*\nu} + w_t^* L x_t^{*\nu}$ ($\forall \nu \in \mathcal{N}$) and $p_t^* b = \pi_t^* p_t^* \omega^b + w_t^* v b$. Then, we obtain

$$\begin{aligned}
L x_t^{*\nu} \gtreqless vb &\Leftrightarrow \pi_t^\nu p_t^* \omega_{t+1}^{*\nu} \lesseqgtr \pi_t^* p_t^* \omega^b \\
&\Leftrightarrow (1+r_t^*)\, p_{t-1}^*\omega_t^{*\nu} - p_t^*\omega_{t+1}^{*\nu} \lesseqgtr (1+r_t^*)\, p_{t-1}^*\omega^b - p_t^*\omega^b \\
&\Leftrightarrow (1+r_t^*)\, p_{t-1}^*\omega_{t+1}^{*\nu} - p_t^*\omega_{t+1}^{*\nu} \lesseqgtr (1+r_t^*)\, p_{t-1}^*\omega^b - p_t^*\omega^b \\
&\Leftrightarrow \left[(1+r_t^*)\, p_{t-1}^*(I-A) - w_t^* L\right]\omega_{t+1}^{*\nu} \lesseqgtr \left[(1+r_t^*)\, p_{t-1}^*(I-A) - w_t^* L\right]\omega^b.
\end{aligned}$$

Therefore, since

$$\sum_{\nu \in \mathcal{N}} \left[(1+r_t^*)\, p_{t-1}^*(I-A) - w_t^* L\right]\omega_{t+1}^{*\nu} = \left[(1+r_t^*)\, p_{t-1}^*(I-A) - w_t^* L\right]\omega^{Nb},$$

we can see that, for each $t = 1, 2, \ldots$, there exists $\nu \in \mathcal{N}$ such that $(1+r_t^*)\, p_{t-1}^*\omega_t^{*\nu} - p_t^*\omega_{t+1}^{*\nu} \neq (1+r_t^*)\, p_{t-1}^*\omega^b - p_t^*\omega^b$ if and only if there exist $\nu, \mu \in \mathcal{N}$ such that $L x_t^{*\nu} > vb$ and $L x_t^{\mu} < vb$, where the latter implies that this IRS is inegalitarian at period t. □

Theorem 1 establishes that in IRSs, a nation is exploiting if and only if its interest income is higher than the world average interest income, and similarly, a nation is exploited if and only if its interest income is lower than the world average.

3.3 The impossibility of the persistent UE

This subsection examines the persistent generation of UE, i.e., it checks whether an RS being inegalitarian at any finite period involves UE even in the infinite limit. First, the following theorem establishes that UE generated in an IRS necessarily tends to disappear in the infinite limit:

Theorem 2: Let $\left\langle \mathcal{N}, (A, L), u, (\omega_1^{\nu})_{\nu \in \mathcal{N}} \right\rangle$ be an economy with $\overline{\omega} = \omega^{Nb}$. For $p_0^* \geqq p_0^* A + w_0^* L$, let $\left\{ p_{t-1}^*; (w_t^*, r_t^*) \right\}_{t=1}^{\infty} = \mathbf{p}\left(p_0^*, w_0^*, \mathbf{r}^*\right)$ be an incompletely specialized IRS with $\left\{ (x_t^{*\nu}, A x_t^{*\nu})_{\nu \in \mathcal{N}} \right\}_{t=1}^{\infty}$. Then, $\lim_{t \to \infty} L x_t^{*\nu} = vb$ holds for any $\nu \in \mathcal{N}$.

Proof. Fix $t = 1, 2, \ldots,$ arbitrarily. Since $p_t^* = \left(1 + r_t^*\right) p_{t-1}^* A + w_t^* L$ and $w_t^* = \left(1 + r_t^*\right) w_{t-1}^*$, we have from $p_t^* \left(I - A\right) x_t^{*\nu} = p_t^* b$ $(\forall \nu \in \mathcal{N})$ that

$$\prod_{\tau=1}^{t} \left(1 + r_\tau^*\right) w_0^* L x_t^{*\nu} = p_t^* b - \left[\left(1 + r_t^*\right) p_{t-1}^* - p_t^*\right] A x_t^{*\nu} \leqq p_t^* b.$$

Moreover, since $p_t^* = \prod_{\tau=1}^{t} \left(1 + r_\tau^*\right) p_0^* A^t + \prod_{\tau=1}^{t} \left(1 + r_\tau^*\right) w_0^* L \sum_{\tau=0}^{t} A^{\tau}$,

$$L x_t^{*\nu} \leqq \frac{p_t^* b}{\prod_{\tau=1}^{t} \left(1 + r_\tau^*\right) w_0^*} = \frac{p_0^*}{w_0^*} A^t b + L \sum_{\tau=0}^{t} A^{\tau} b, \quad (\forall t). \qquad (10)$$

Note that $A^t \to \mathbf{0}$ as $t \to \infty$. Then, the first term on the rightmost side of (10), $\frac{p_0^*}{w_0^*} A^t b$, converges to 0 in the infinite limit of time. Also, the second term has the property, $\lim_{t \to \infty} L \sum_{\tau=0}^{t} A^{\tau} b = L \left(I - A\right)^{-1} b = vb$. Thus, (10) implies $\lim_{t \to \infty} L x_t^{*\nu} \leqq vb$. Moreover, such an allocation attains as an RS given that $\lim_{t \to \infty} L x_t^{\nu} = vb$. Indeed, if $\lim_{t \to \infty} L x_t^{*\nu} < vb$ for some $\nu \in \mathcal{N}$, then $\lim_{t \to \infty} L x_t^{\mu} > vb$ for some $\mu \in \mathcal{N} \setminus \{\nu\}$, which violates the feasibility condition for μ. □

What is the underlying mechanism to establish the impossibility of persistent UE? The following observation of an equilibrium price system $\mathbf{p}\left(p_0, w_0, \mathbf{r}\right)$ under the IRS gives us useful information:

Proposition 5: For $\mathbf{r} \subset \overline{\mathbb{R}}_+$, let $\left\{ p_{t-1}; (w_t, r_t,) \right\}_{t=1}^{\infty} = \mathbf{p}\left(p_0, w_0, \mathbf{r}\right)$ with $p_0 \geq p_0 A + w_0 L$. Then, the followings holds:

(i) if $\limsup_{t\to\infty} r_t < R$, then $\lim_{t\to\infty}[(1+r_t)p_{t-1}-p_t] = \mathbf{0}$; and conversely,

(ii) if $\liminf_{t\to\infty} r_t > R$, then $\liminf_{t\to\infty}[(1+r_t)p_{t-1}-p_t]_i > 0$ for all i.

Proof. See Appendix. □

Proposition 5 classifies the equilibrium price systems of incompletely specialized RSs, according to whether the nominal interest rates in the infinite limit remains above or below the threshold. In the former case, the real interest rate converges to zero: $\lim_{t\to\infty}[(1+r_t)p_{t-1}-p_t] = \mathbf{0}$, while in the latter, it remains positive.

Remark 1: Proposition 5 suggests that an equilibrium price system of an incompletely specialized RS does not necessarily converge to a stationary system of relative prices of commodities. Indeed, it can be shown that there exists an incompletely specialized RS in which the sequence of equilibrium interest rates is such that $\liminf_{t\to\infty} r_t > R$.[17]

The classification by Proposition 5 is key for characterizing the mechanism to make the generated UE disappear in the infinite limit. First, we have.

Corollary 3: Let $\left\langle \mathcal{N}, (A,L), u, (\omega_1^\nu)_{\nu\in\mathcal{N}} \right\rangle$ be an economy with $\overline{\omega} = \omega^{Nb}$. For $p_0^* \geqq p_0^*A + w_0^*L$, let $\left\{p_{t-1}^*; (w_t^*, r_t^*)\right\}_{t=1}^\infty = \mathbf{p}(p_0^*, w_0^*, \mathbf{r}^*)$ be an incompletely specialized IRS with $\left\{(x_t^{*\nu}, Ax_t^{*\nu})_{\nu\in\mathcal{N}}\right\}_{t=1}^\infty$ such that $\limsup_{t\to\infty} r_t^* < R$. Then, $\lim_{t\to\infty} Lx_t^{*\nu} = vb$ holds for any $\nu\in\mathcal{N}$, while $\lim_{t\to\infty}\left[(1+r_t^*)p_{t-1}^*\omega_t^{*\nu} - p_t^*\omega_{t+1}^{*\nu}\right] = 0$ for any $\nu\in\mathcal{N}$.

Proof. By Proposition 5(i), $\lim_{t\to\infty}\left[(1+r_t^*)p_{t-1}^* - p_t^*\right] = \mathbf{0}$ holds. Then, for any $\nu\in\mathcal{N}$,

$$\begin{aligned}\lim_{t\to\infty}\left[(1+r_t^*)p_{t-1}^*\omega_t^{*\nu} - p_t^*\omega_{t+1}^{*\nu}\right] &= \lim_{t\to\infty}\left[(1+r_t^*)p_{t-1}^* - p_t^*\right]\omega_{t+1}^{*\nu} \\ &= 0 = \lim_{t\to\infty}\left[(1+r_t^*)p_{t-1}^* - p_t^*\right]\omega^b,\end{aligned}$$

by $p_{t-1}^*\omega_t^{*\nu} = p_{t-1}^*Ax_t^{*\nu} = p_{t-1}^*\omega_{t+1}^{*\nu}$. Then, by Theorem 1, it concludes that $\lim_{t\to\infty} Lx_t^{*\nu} = vb$ for any $\nu\in\mathcal{N}$. □

[17] See the Addendum for a detailed discussion of it.

Thus, Corollary 3 shows that the generated UE tends to disappear in the infinite limit while, equivalently, the real interest income converges to zero. This result is similar to Veneziani's (2007) result, which shows that under an economy with a labor market, UE tends to disappear in the infinite limit, due to the tendency of the profit rates to converge to zero.

However, an IRS does not necessarily fulfill $\lim_{t\to\infty}\left[(1+r_t^*)\,p_{t-1}^* - p_t^*\right] = \mathbf{0}$, so that the real interest rate does not necessarily converge to zero. In this case, the argument underlying Corollary 3 cannot apply. Even in such cases, an IRS tends to equalize the associated allocations across nations and eventually attains no UE. In fact:

Corollary 4: Let $\left\langle \mathcal{N}, (A,L), u, (\omega_1^\nu)_{\nu\in\mathcal{N}} \right\rangle$ be an economy with $\overline{\omega} = \omega^{Nb}$. For $p_0^* \geq p_0^* A + w_0^* L$, let $\left\{p_{t-1}^*; (w_t^*, r_t^*)\right\}_{t=1}^{\infty} = \mathbf{p}\left(p_0^*, w_0^*, \mathbf{r}^*\right)$ be an incompletely specialized IRS with $\left\{(x_t^{*\nu}, Ax_t^{*\nu})_{\nu\in\mathcal{N}}\right\}_{t=1}^{\infty}$ such that $\liminf_{t\to\infty} r_t^* > R$. Then, $\lim_{t\to\infty} Lx_t^{*\nu} = vb$ holds for any $\nu \in \mathcal{N}$, while equivalently $\lim_{t\to\infty}\omega_{t+1}^\nu = \omega^b$ $(\forall \nu \in \mathcal{N})$, i.e., the wealth distribution converges to be egalitarian in the infinite limit.

Proof. By Proposition 5(ii), $\lim_{t\to\infty}\left[(1+r_t^*)\,p_{t-1}^* - p_t^*\right] > \mathbf{0}$ holds. Then, as $p_0^* \geq p_0^* A + w_0^* L$ and $p_0^* A$ and L are linearly independent, p_t^* does not converge to be labor value pricing by Proposition 3(ii). Equivalently, $p_t^*(I-A)$ and L do not converge to be linearly dependent.

Let $Lx^{*\nu} \equiv \lim_{t\to\infty} Lx_t^{*\nu}$ for any $\nu \in \mathcal{N}$. By Theorem 2, $Lx^{*\nu} = Lx^{*\mu}$ and $p_t^*(I-A)\,x_t^{*\nu} = p_t^* b = p_t^*(I-A)\,x_t^{*\mu}$ for any t. Suppose that $x^{*\nu} \neq x^{*\mu}$ for some $\nu, \mu \in \mathcal{N}$. Then, $p_t^*(I-A)\cdot(x_t^{*\nu} - x_t^{*\mu}) = 0$ for any t and $\lim_{t\to\infty} L\,(x_t^{*\nu} - x_t^{*\mu}) = 0$ with $x^{*\nu} \neq x^{*\mu}$ imply that $p_t^*(I-A)$ and L converge to be linearly dependent. Therefore, p_t^* converges to be labor value pricing as $t \to \infty$, which is a contradiction. Thus, $x^{*\nu} = x^{*\mu}$ holds for any $\nu, \mu \in \mathcal{N}$. This implies that $\lim_{t\to\infty} x_t^{*\nu} = (I-A)^{-1} b$ for any $\nu \in \mathcal{N}$. This is equivalent to $\lim_{t\to\infty}\omega_{t+1}^\nu = \lim_{t\to\infty} Ax_t^{*\nu} = A\,(I-A)^{-1} b = \omega^b$ for any $\nu \in \mathcal{N}$. □

Remark 2: All of the main theorems in this paper can be preserved even if the overtaking criterion is applied instead of the catching up criterion.

Remark 3: Though the result of Corollary 3 looks similar to the impossibility of persistent exploitation in Venezaini (2007), there are some significant differences between these two. The model developed in Veneziani (2007) is an approximation of a continuous time model, in that prices of commodities at the beginning of a production period are assumed to be identical to those at the end of that period. Because of this setting, an equilibrium price system in Veneziani (2007) can be characterized by: $p_t = (1 + r_t) p_t A + L$ and $p_t = (1 + r_{t+1}) p_{t+1}$ for every $t = 1, 2, \ldots$. To preserve such a property, v and L must be the Frobenius eigenvectors of A, and the commodity price vector p must be labor value pricing. Under such a specific environment, $\lim_{t\to\infty} r_t = 0$ holds.[18] In contrast, Corollary 3 of this paper can be applied to a broader class of environments beyond such a specific one.

Remark 4: The type of non-persistency of UE discussed in Corollary 4 may be attributed to an intrinsic feature of economies without (international) markets for labor or credit. For inequality (10), which is critical to derive the non-persistent UE, is established under the assumption of $p_t^* (I - A) x_t^{*\nu} = p_t^* b$ $(\forall \nu \in \mathcal{N})$ at every period t. This condition holds as a feature of RSs only in economies without any (international) factor market. Indeed, if we assume a model of economies with labor markets like Veneziani (2007), then there may exist agents in such a model, who have no endowment of material input goods but can survive as employed workers. Since such an agent does not satisfy $p_t^* (I - A) x_t^{*\nu} = p_t^* b$, Theorem 2 and Corollary 4 cannot apply to such a model of economies.

4 Concluding Remarks

This paper presented a model of infinite-horizon international economies with a leisure preference and a Leontief production technique and characterized free trade equilibria, RSs, in terms of the persistency of UE. First, focusing on incompletely specialized RSs where factor price equalization holds, the necessary and sufficient condition for the emergence of UE was examined (Theorem 1). Moreover, it was shown that the UE that emerged in such equilibria tends to disappear in the infinite limit (Theorem 2). Two mechanisms driving the observed UE to vanish were discussed. One mechanism corresponds to the subclass of equilibria whose associated sequences of equilibrium prices converge to the labor-value pricing. In such equilibria, the UE

[18] More precisely speaking, $p = \lim_{t\to\infty} \prod_{\tau=1}^{t} (1 + r_\tau) v$ holds, which also implies that $\lim_{t\to\infty} r_t = 0$ should hold to ensure $\lim_{t\to\infty} \prod_{\tau=1}^{t} (1 + r_\tau) < \infty$.

that emerges tends to disappear because the sequences of equilibrium interest rates converge to zero (Corollary 3). The other mechanism corresponds to the subclass of equilibria in which the distributions of capital endowments tend to be equalized in the long run. Because of this equalization, UE cannot emerge persistently (Corollary 4). Finally, if a RS does not preserve factor price equalization, then its equilibrium prices are the labor-value pricing, and so no UE emerges.

Among these results, the second type of mechanism for non-persistent UE is particularly interesting. First, such a mechanism corresponds to the case where the equilibrium prices do not converge to any stationary price vector and the equilibrium 'real' interest rates do not converge to zero. Neither the standard literature of intertemporal general equilibrium theory nor Marxian exploitation theory has paid attention to this case. Such a mechanism is not vacuous, since these types of RSs can exist under a reasonable restriction of economic environments, as shown in the Addendum. Second, the functioning of such a mechanism may be attributed to an intrinsic feature of economies with neither (international) labor nor credit markets. The tendency of the distribution of capital endowments to equalize, proved in Corollary 4, relies crucially on the property that neither labor nor credit markets are internationalized. Indeed, this equalization tendency is not necessarily observed if either (international) labor or credit markets exist, according to Chen and Yoshihara (2018).[19]

A relevant work was developed by Veneziani (2007), who also assumes an intertemporal economy with a labor market, the leisure preference, and no time preference, and then shows that UE tends to vanish in the infinite limit.[20] However, in his model, no time structure of production like ours is assumed, and so at every period any equilibrium commodity prices are labor-value pricing and converge to the (absolute) labor value, as mentioned by Remark 3 in section 3.3 above. In contrast, in a model with a time structure of production like ours, the same two types of equilibrium sequences of prices

[19] Chen and Yoshihara (2018) show that, in a perfectly competitive economy with a labor market and the leisure preference, any interior RS is associated with a sequence of equilibrium (real) interest rates converging to zero, whenever propertyless agents of the working class exist in the initial period.

[20] If the leisure preference is replaced by the standard strongly monotonic preferences in the intertemporal economy with the labor market and no time preference, then it is even easy to show that UE tends to vanish in the finite period of time. (See Giorgos, Veneziani, and Yoshihara (2018).)

(as shown in Proposition 5) can be observed even in economies with a labor or credit market. Then, it can be shown, due to Chen and Yoshihara (2018), that UE emerges at every period persistently if the equilibrium sequence of profit rates or interest rates does not converge to zero. It is in a sharp contrast with Corollary 4 of this paper obtained under the international trade model.

5 Appendix

Proof of Proposition 1:

Proof. Let $\{x_t^\nu\}_{t=1}^\infty$ be catching-up optimal. Note that the inequality constraint (3) holds with equality under optimality. Suppose on the contrary that $p_\tau x_\tau^\nu + p_\tau \delta_\tau^\nu > p_\tau b + p_\tau \omega_{\tau+1}^\nu$ for some τ. Then, by the assumption of $x_t^\nu \geq \mathbf{0}$ $(t = 1, 2, \ldots)$, we can take another possible path $\{x_t'\}_{t=1}^\infty \in \mathbf{x}(\omega_1^\nu)$ such that $x_\tau' \leq x_\tau^\nu$ for τ and $x_t' = x_t^\nu$ for $t = 1, 2, \ldots, \tau-1, \tau+1, \ldots$. We have from $L > \mathbf{0}$ that

$$\liminf_{T\to\infty} \sum_{t=1}^{T} (Lx_t' - Lx_t^\nu) = (Lx_\tau' - Lx_\tau^\nu) < 0,$$

which contradicts the optimality. Then, the following holds:

$$\begin{aligned} w_t^\nu L x_t^\nu &= p_t x_t^\nu - (1 + r_t^\nu)\, p_{t-1} A x_t^\nu \quad \text{by (4)} \\ &= p_t b + p_t \omega_{t+1}^\nu - p_t \delta_t^\nu - (1 + r_t^\nu)\, p_{t-1} A x_t^\nu \quad \text{by (3) with equality} \\ &= p_t b + p_t \omega_{t+1}^\nu - (1 + r_t^\nu)\, p_{t-1} (\omega_t^\nu - \delta_t^\nu) - p_t \delta_t^\nu \quad \text{by (1)}. \end{aligned}$$

Putting $s_t^\nu \equiv \omega_{t+1}^\nu - \omega_t^\nu$, we have, from $w_t^\nu > 0$ for any $\nu \in \mathcal{N}$, that

$$Lx_t^\nu = \frac{p_t b + p_t s_t^\nu + [p_t - (1 + r_t^\nu)\, p_{t-1}]\, (\omega_t^\nu - \delta_t^\nu)}{w_t^\nu},$$

and analogically,

$$Lx_{t+1}^\nu = \frac{p_{t+1} b + p_{t+1} s_{t+1}^\nu + \left[p_{t+1} - \left(1 + r_{t+1}^\nu\right) p_t\right] \left(\omega_{t+1}^\nu - \delta_{t+1}^\nu\right)}{w_{t+1}^\nu}.$$

As a one-period perturbation for an arbitrary $\tau < \infty$, let $s_\tau' \equiv s_\tau^\nu + \Delta_\tau^\nu$ and $s_{\tau+1}' \equiv s_{\tau+1}^\nu + \Delta_{\tau+1}^\nu$ meet $\Delta_\tau^\nu = -\Delta_{\tau+1}^\nu$. In the perturbation path,

$$Lx_\tau' + Lx_{\tau+1}' = Lx_\tau^\nu + Lx_{\tau+1}^\nu + \left[\frac{p_\tau \Delta_\tau^\nu}{w_\tau^\nu} - \frac{(1 + r_{\tau+1}^\nu) p_\tau \Delta_\tau^\nu}{w_{\tau+1}^\nu}\right].$$

Since $\{x_t^\nu\}_{t=1}^\infty$ is catching-up optimal, $\frac{p_\tau \Delta_\tau^\nu}{w_\tau^\nu} \geqq \frac{(1+r_{\tau+1}^\nu) p_\tau \Delta_\tau^\nu}{w_{\tau+1}^\nu}$ holds. Indeed, if $\frac{p_\tau \Delta_\tau^\nu}{w_\tau^\nu} < \frac{(1+r_{\tau+1}^\nu) p_\tau \Delta_\tau^\nu}{w_{\tau+1}^\nu} < 0$, then

$$\liminf_{T\to\infty} \sum_{t=1}^{T} \left(Lx_t' - Lx_t^\nu\right) = \left[\frac{p_\tau \Delta_{\tau-1}^\nu}{w_\tau^\nu} - \frac{\left(1 + r_{\tau+1}^\nu\right) p_\tau \Delta_\tau^\nu}{w_{\tau+1}^\nu}\right] < 0,$$

which contradicts the optimality of $\{x_t^\nu\}_{t=1}^\infty$. Note the assumption of $p_{t-1}, p_t > \mathbf{0}$ for all $t = 1, 2, \ldots$. Therefore, if $\frac{p_{i\tau}}{w_\tau^\nu} > \frac{\left(1+r_{\tau+1}^\nu\right) p_{i\tau}}{w_{\tau+1}^\nu}$ for some commodity i, then there is a suitable perturbation $\Delta_{i\tau}^\nu < 0$ such that the perturbation path is possible. This generates $Lx_\tau' + Lx_{\tau+1}' < L_\tau^\nu + Lx_{\tau+1}^\nu$, which contradicts the optimality by an argument similar to the one above. Since we take $\tau < \infty$ arbitrarily, we obtain that for each $t = 1, 2 \ldots$, $\frac{p_t}{w_t^\nu} = \frac{\left(1+r_{t+1}^\nu\right) p_t}{w_{t+1}^\nu}$, which yields $w_{t+1}^\nu = \left(1 + r_{t+1}^\nu\right) w_t^\nu$ by $p_t \geq \mathbf{0}$. This completes the proof. □

Proof of Proposition 2:

Proof. (i) Note that an incompletely specialized RS yields $p_t^* = (1 + r_t^{\nu *})\, p_{t-1}^* A + w_t^{\nu *} L$ for every $\nu \in \mathcal{N}$. Fix arbitrarily $\nu, \mu \in \mathcal{N}$. Then, we obtain from $(1 + r_t^{\nu *})\, p_{t-1}^* A + w_t^{\nu *} L = p_t^* = (1 + r_t^{\mu *})\, p_{t-1}^* A + w_t^{\mu *} L$ that $(r_t^{\nu *} - r_t^{\mu *})\, p_{t-1}^* A + (w_t^{\nu *} - w_t^{\mu *})\, L = \mathbf{0}$. This implies that for i and j such that $i \neq j$ and $\frac{\left(p_{t-1}^* A\right)_i}{L_i} \neq \frac{\left(p_{t-1}^* A\right)_j}{L_j}$,

$$\left(r_t^{\nu *} - r_t^{\mu *}, w_t^{\nu *} - w_t^{\mu *}\right) \begin{bmatrix} p_{t-1}^* A e_i & p_{t-1}^* A e_j \\ L_i & L_j \end{bmatrix} = (0, 0)\,.$$

Since the matrix

$$\begin{bmatrix} p_{t-1}^* A e_i & p_{t-1}^* A e_j \\ L_i & L_j \end{bmatrix}$$

is nonsingular by $\frac{\left(p_{t-1}^* A\right)_i}{L_i} \neq \frac{\left(p_{t-1}^* A\right)_j}{L_j}$, we obtain $(w_t^{\nu *}, r_t^{\nu *}) = (w_t^{\mu *}, r_t^{\mu *})$. Note that this result follows for every $\nu, \mu \in \mathcal{N}$, which completes the proof.

(ii) Let $(w_t^{\nu *}, r_t^{\nu *}) = (w_t^{\mu *}, r_t^{\mu *}) = (w_t^*, r_t^*)$ for every $\nu, \mu \in \mathcal{N}$. If $p_t^* A$ and L are linearly independent, then $\left(w_{t+1}^{\nu *}, r_{t+1}^{\nu *}\right) = \left(w_{t+1}^{\mu *}, r_{t+1}^{\mu *}\right) = \left(w_{t+1}^*, r_{t+1}^*\right)$ for every $\nu, \mu \in \mathcal{N}$ by (i). Therefore, suppose that $p_t^* A$ and L are linearly dependent. This implies that

$$\frac{p_{it+1}^*}{p_{jt+1}^*} = \frac{p_t^* A e_i}{p_t^* A e_j} = \frac{L_i}{L_j} \text{ for all } i \text{ and } j.$$

In such a case, for any nation $\nu \in \mathcal{N}$, any $\left(w_{t+1}^{\nu *}, r_{t+1}^{\nu *}\right) \in \mathbb{R}_{+}^{2} \setminus \{\mathbf{0}\}$ can be the factor price vectors whenever $p_{t+1}^{*} = \left(1 + r_{t+1}^{\nu *}\right) p_{t}^{*} A + w_{t+1}^{\nu *} L$ holds. Suppose that $\left(w_{t+1}^{\nu *}, r_{t+1}^{\nu *}\right) \neq \left(w_{t+1}^{\mu *}, r_{t+1}^{\mu *}\right)$ for some $\nu, \mu \in \mathcal{N}$. First, let $r_{t+1}^{\nu *} > r_{t+1}^{\mu *}$ without loss of generality. Then, by Proposition 1, it follows that

$$w_{t+1}^{\nu *} = \left(1 + r_{t+1}^{\nu *}\right) w_{t}^{*} > \left(1 + r_{t+1}^{\mu *}\right) w_{t}^{*} = w_{t+1}^{\mu *}.$$

Then,

$$\begin{aligned} p_{t+1}^{*} &= \left(1 + r_{t+1}^{\nu *}\right) p_{t}^{*} A + w_{t+1}^{\nu *} L = \left(1 + r_{t+1}^{\nu *}\right) p_{t}^{*} A + \left(1 + r_{t+1}^{\nu *}\right) w_{t}^{*} L \\ &> \left(1 + r_{t+1}^{\mu *}\right) p_{t}^{*} A + \left(1 + r_{t+1}^{\mu *}\right) w_{t}^{*} L = \left(1 + r_{t+1}^{\mu *}\right) p_{t}^{*} A + w_{t+1}^{\mu *} L = p_{t+1}^{*}, \end{aligned}$$

which is a contradiction. Therefore, $r_{t+1}^{\nu *} = r_{t+1}^{\mu *}$ must hold. Second, let $w_{t+1}^{\nu *} > w_{t+1}^{\mu *}$ without loss of generality. However, by Proposition 1, this inequality implies that $r_{t+1}^{\nu *} > r_{t+1}^{\mu *}$, which is a contradiction. Therefore, $\left(w_{t+1}^{\nu *}, r_{t+1}^{\nu *}\right) = \left(w_{t+1}^{\mu *}, r_{t+1}^{\mu *}\right) = \left(w_{t+1}^{*}, r_{t+1}^{*}\right)$ holds for every $\nu, \mu \in \mathcal{N}$, whenever $\left(w_{t}^{\nu *}, r_{t}^{\nu *}\right) = \left(w_{t}^{\mu *}, r_{t}^{\mu *}\right) = \left(w_{t}^{*}, r_{t}^{*}\right)$ for every $\nu, \mu \in \mathcal{N}$. □

Proof of Proposition 3:

Proof. Let $\{p_{t-1}; (w_t, r_t)\}_{t=1}^{\infty} = \mathbf{p}(p_0, w_0, \mathbf{r})$. We notice that $w_t = \prod_{\tau=1}^{t} (1 + r_\tau) w_0$ for all $t = 1, 2, \ldots$. Then, we obtain that for any $t = 1, 2, \ldots$,

$$(1 + r_t) p_{t-1} - p_t = \prod_{\tau=1}^{t} (1 + r_\tau) \left[p_0 (I - A) - w_0 L\right] A^{t-1} \geqq \mathbf{0} \qquad (7)$$

by the induction on t. Indeed,

$$(1 + r_1) p_0 - p_1 = (1 + r_1) p_0 - (1 + r_1) p_0 A - w_0 L = (1 + r_1) \left[p_0 (I - A) - w_0 L\right]$$

holds and supposing

$$(1 + r_t) p_{t-1} - p_t = \prod_{\tau=1}^{t} (1 + r_\tau) \left[p_0 (I - A) - w_0 L\right] A^{t-1}$$

yields

$$
\begin{aligned}
&(1+r_{t+1})\,p_t - p_{t+1} \\
&= (1+r_{t+1})\left[(1+r_t)\,p_{t-1}A + \prod_{\tau=1}^{t}(1+r_\tau)\,w_0L\right] - (1+r_{t+1})\,p_tA - \prod_{\tau=1}^{t+1}(1+r_\tau)\,w_0L \\
&= (1+r_{t+1})\left[(1+r_t)\,p_{t-1} - p_t\right]A \\
&= \prod_{\tau=1}^{t+1}(1+r_\tau)\left[p_0\,(I-A) - w_0L\right]A^t.
\end{aligned}
$$

For (i),

$$\mathbf{0} \leq (1+r_1)\,p_0 - p_1 = (1+r_1)\,(p_0 - p_0A - w_0L)\,,$$

which proves the "only if" part. For the "if" part of (i), let $p_0 \geq p_0A + w_0L$. Then, $(1+r_1)\,p_0 - p_1 \geq \mathbf{0}$. Inductively, suppose $(1+r_t)\,p_{t-1} - p_t \geq \mathbf{0}$. Note that (7) yields

$$(1+r_{t+1})\,p_t - p_{t+1} = (1+r_{t+1})\left[(1+r_t)\,p_{t-1} - p_t\right]A. \qquad (8)$$

Then, the indecomposability of A ensures $(1+r_{t+1})\,p_t - p_{t+1} \geq \mathbf{0}$ (see footnote 6), which completes the proof of (i).

For (ii), it suffices to show the "only if" part since the "if" part is obvious. Let $q > \mathbf{0}$ and $\varsigma_t > 0$ satisfy $p_t = \varsigma_t q$ $(t = 0, 1, \ldots)$. Fixing t arbitrarily, we have $(1+r_{t+1})\,p_t - p_{t+1} = \left[(1+r_{t+1})\,\varsigma_t - \varsigma_{t+1}\right]q$. Since $(1+r_{t+1})\,p_t - p_{t+1} = (1+r_{t+1})\left[(1+r_t)\,p_{t-1} - p_t\right]A$ by (7), we derive

$$\left[(1+r_{t+1})\,\varsigma_t - \varsigma_{t+1}\right]q = (1+r_{t+1})\left[(1+r_t)\,\varsigma_{t-1} - \varsigma_t\right]qA.$$

Then, $(1+r_t)\,\varsigma_{t-1} - \varsigma_t = 0$ holds for all $t = 1, 2, \ldots$; or $q > \mathbf{0}$ is an eigenvector for A, and therefore, must be the unique (up to scalar) Perron-Frobenius eigenvector by the indecomposability of A. For the former case, it is equivalent to

$$p_0 = p_0A + w_0L \;\Leftrightarrow\; p_0 = w_0v$$

by (7), so that $q = \frac{w_0}{\varsigma_0}v$. For the latter case, let $\frac{1}{1+R}$ be the Perron-Frobenius eigenvalue of A. Note that L must be the Perron-Frobenius eigenvector of A

as well, since

$$\begin{aligned}
\varsigma_1 q = p_1 &= (1+r_1)\, p_0 A + (1+r_1)\, w_0 L \\
&= \frac{1+r_1}{1+R} p_0 + (1+r_1)\, w_0 L \\
&= \frac{1+r_1}{1+R} \varsigma_0 q + (1+r_1)\, w_0 L \\
\Leftrightarrow \;\; L &= \frac{(1+R)\,\varsigma_1 - (1+r_1)\,\varsigma_0}{(1+r_1)\,(1+R)\, w_0} q.
\end{aligned}$$

Then, from the Neumann series $(I-A)^{-1} = \sum_{\tau=0}^{\infty} A^{\tau}$, we have that

$$v = L + LA + LA^2 + \cdots = L + \frac{1}{1+R} L + \left(\frac{1}{1+R}\right)^2 L + \cdots = \frac{1+R}{R} L,$$

and therefore, $q = \frac{(1+r_1)Rw_0}{(1+R)\varsigma_1 - (1+r_1)\varsigma_0} v$. □

Proof of Proposition 4:

Proof. We first show that, under $\overline{\omega} = \omega^{Nb}$, there only exists an RS such that $\omega_t^* \geqq \omega^{Nb}$ for all $t = 1, 2, \ldots$. Suppose on the contrary that $\omega_t^* \ngeqq \omega^{Nb}$ for some t. Fixing such t and taking $\varepsilon \equiv \omega_t^* - \omega^{Nb}$, we have $\varepsilon_i < 0$ for some i. To derive a contradiction, it suffices to focus on exhaustive production processes, that is, $\left\{ \left(x_t^{*\nu}, \delta_t^{*\nu}, \omega_{t+1}^{*\nu} \right)_{\nu \in \mathcal{N}} \right\}_{t=1}^{\infty}$ fulfilling $Ax_t^* = \omega_t^* - \delta_t^*$ and $\omega_{t+1}^* =$

$(I-A)\,x_t^* + \omega_t^* - Nb = x_t^* + \delta_t^* - Nb$ for all $t = 1, 2, \ldots$. Then, for t fixed,

$$\begin{aligned}
A^\tau \omega_{t+\tau}^* &= A^\tau \left(x_{t+\tau-1}^* + \delta_{t+\tau-1}^*\right) - A^\tau (Nb) = A^{\tau-1}\omega_{t+\tau-1}^* - A^{\tau-1}(I-A)\,\delta_{t+\tau-1}^* - A^\tau (Nb) \\
&= A^{\tau-1}\left(x_{t+\tau-2}^* + \delta_{t+\tau-2}^*\right) - A^{\tau-1}(Nb) - A^{\tau-1}(I-A)\,\delta_{t+\tau-1}^* - A^\tau (Nb) \\
&= A^{\tau-2}\omega_{t+\tau-2}^* - A^{\tau-1}(I+A)(Nb) - A^{\tau-1}(I-A)\,\delta_{t+\tau-1}^* - A^{\tau-2}(I-A)\,\delta_{t+\tau-2}^* \\
&= \cdots = \omega_t^* - A\left(I + A + \cdots + A^{\tau-1}\right)(Nb) - \sum_{k=0}^{\tau-1} A^k (I-A)\,\delta_{t+k}^* \\
&= \omega^{Nb} - A\left(I + A + \cdots + A^{\tau-1}\right)(Nb) + \varepsilon - \sum_{k=0}^{\tau-1} A^k (I-A)\,\delta_{t+k}^* \\
&= A\left(I + A + A^2 + \cdots\right)(Nb) - A\left(I + A + \cdots + A^{\tau-1}\right)(Nb) + \varepsilon - \sum_{k=0}^{\tau-1} A^k (I-A)\,\delta_{t+k}^* \\
&= A^{\tau+1}\left(I + A + A^2 + \cdots\right)(Nb) + \varepsilon = A^\tau \omega^{Nb} + \varepsilon - \sum_{k=0}^{\tau-1} A^k (I-A)\,\delta_{t+k}^*.
\end{aligned}$$

Since $(I-A)$ is nonsingular, $A^t \to \mathbf{0}$ as $t \to \infty$, so that there exists a sufficiently large τ such that $\left(A^\tau \omega^{Nb}\right)_i + \varepsilon_i < 0$ for i concerned here. Then, as $\sum_{k=0}^{\tau-1} A^k (I-A) \doteqdot (I-A)^{-1}(I-A)$ for sufficiently large τ, we have $\sum_{k=0}^{\tau-1} A^k (I-A)\,\delta_{t+k}^* \geqq \sum_{k=0}^{\tau-1} A^k (I-A)\,\delta^*(\tau)$ where $\delta^*(\tau) \equiv \left(\delta_j^*(\tau)\right)_{j=1,\ldots,n}$ with $\delta_j^*(\tau) \equiv \min_{0 \leqq h \leqq \tau-1}\left\{\delta_{jt+h}^*\right\}$ for each commodity $j = 1, \ldots, n$. Thus, $A^\tau \omega_{t+\tau}^* = A^\tau \omega^{Nb} + \varepsilon - \sum_{k=0}^{\tau-1} A^k (I-A)\,\delta_{t+k}^* \leqq A^\tau \omega^{Nb} + \varepsilon - \sum_{k=0}^{\tau-1} A^k (I-A)\,\delta^*(\tau)$ and $-\sum_{k=0}^{\tau-1} A^k (I-A)\,\delta^* \doteqdot -\delta^*(\tau)$ for the sufficiently large τ. Therefore, $A^\tau \omega_{t+\tau}^* \leqq A^\tau \omega^{Nb} + \varepsilon - \sum_{k=0}^{\tau-1} A^k (I-A)\,\delta^*(\tau)$ and $\left(A^\tau \omega^{Nb}\right)_i + \varepsilon_i - \delta_i^*(\tau) < 0$. This implies $\omega_{t+\tau}^* \notin \mathbb{R}_+^n$, which is a contradiction.

We next confirm that conditions (b) and (c) of Definition 1 hold for $t = 1$ if and only if $x_1^* = (I-A)^{-1}(Nb)$. Indeed, condition (b) implies $x_1^* + \delta_1^* \geqq Nb + \omega_2^*$, which implies $(I-A)\,x_1^* + \omega^{Nb} \geqq Nb + \omega_2^* \geqq Nb + \omega^{Nb}$ by $\overline{\omega} = \omega^{Nb}$ and $\omega_2^* \geqq \omega^{Nb}$. Therefore, $(I-A)\,x_1^* \geqq Nb$, which implies $x_1^* \geqq (I-A)^{-1}(Nb)$. Moreover, if $x_1^* \geq (I-A)^{-1}(Nb)$, then $Ax_1^* \geq A(I-A)^{-1}(Nb) = \omega^{Nb}$ by the indecomposability of A. Thus, it violates condition (c) by $\overline{\omega} = \omega^{Nb}$. Therefore, $x_1^* = (I-A)^{-1}(Nb)$ must hold.

Moreover $x_1^* = (I-A)^{-1}(Nb)$ yields $\omega_2^* = \omega^{Nb}$, so that the same argument applies to $x_t^* = (I-A)^{-1}(Nb)$, inductively. In conclusion, any RS allocation meets $x_t^* = (I-A)^{-1}(Nb)$, $\delta_t^* = \mathbf{0}$, and $\omega_{t+1}^* = \omega^{Nb}$ for all $t = 1, 2, \ldots$. □

Proof of Lemma 1:

Proof. Note that $\{(x_t^\nu, \omega_t^\nu)\}_{t=1}^T$ meets all of the constraints in (MP_T^ν) since a catching-up optimal path $\{(x_t^\nu, \delta_t^\nu, \omega_{t+1}^\nu)\}_{t=1}^\infty$ is possible, i.e., satisfies (1)–(4). Suppose on the contrary that $\{(x_t^\nu, \delta_t^\nu, \omega_{t+1}^\nu)\}_{t=1}^T$ is not a solution to (MP_T^ν) for some T. For this T, there exists another solution, $\{(x_t^{\nu\prime}, \delta_t^{\nu\prime}, \omega_{t+1}^{\nu\prime})\}_{t=1}^T$, which implies $\sum_{t=1}^T Lx_t^{\nu\prime} < \sum_{t=1}^T Lx_t^\nu$. Let us take $\{(x_t^{\nu\prime}, \delta_t^{\nu\prime}, \omega_{t+1}^{\nu\prime})\}_{t=T+1}^\infty$ such that $x_t^{\nu\prime} = x_t^\nu$, $\delta_t^{\nu\prime} = \delta_t^\nu$, and $\omega_{t+1}^{\nu\prime} = \omega_{t+1}^\nu$ for $t = T+1, T+2, \ldots$, which is possible by $p_T A x_{T+1}^\nu = p_T \omega_{T+1}^\nu = p_T \omega_{T+1}'$. Then, $\{(x_t^{\nu\prime}, \delta_t^{\nu\prime}, \omega_{t+1}^{\nu\prime})\}_{t=1}^\infty$ catches up to $\{(x_t^\nu, \delta_t^\nu, \omega_{t+1}^\nu)\}_{t=1}^\infty$, i.e.,

$$\liminf_{T\to\infty} \sum_{t=1}^T (Lx_t' - Lx_t^\nu) = \sum_{t=1}^T (Lx_t' - Lx_t^\nu) < 0,$$

which is a contradiction. □

Proof of Lemma 2:

Proof. Since $p_t = (1+r_t)\, p_{t-1} A + w_t L > \mathbf{0}$ and $w_t = (1+r_t)\, w_{t-1}$ for $t = 1, 2, \ldots, T$,

$$w_0 L = \frac{p_T - (1+r_T)\, p_{T-1} A}{\prod_{t=1}^T (1+r_t)} = \frac{p_{T-1} - (1+r_{T-1})\, p_{T-2} A}{\prod_{t=1}^{T-1} (1+r_t)} = \cdots = \frac{p_1 - (1+r_1)\, p_0 A}{1+r_1}.$$

Let $\{(x_t', \delta_t', \omega_{t+1}')\}_{t=1}^T$ be any plan satisfying all of the constraints in (MP_T^ν). Then, constraint (1) for $t = 1$ yields that $p_0 A x_1' + p_0 \delta_1' = p_0 \omega_1^\nu = p_0 A x_1^\nu$. Given that constraint (3) holds with equality, the constraints (1) and (3) for $t = 1, 2, \ldots, T-1$ imply that

$$\begin{aligned} p_t x_t' + p_t \delta_t' &= p_t b + p_t \omega_{t+1}' \\ &= p_t b + p_t A x_{t+1}' + p_t \delta_{t+1}' , \end{aligned}$$

while for $t = T$,

$$\begin{aligned} p_T x_T' + p_T \delta_T' &= p_T b + p_T \omega_{T+1}' \\ &= p_T b + p_T \omega_{T+1}^\nu . \end{aligned}$$

Therefore,

$$\begin{aligned} p_t\left(x'_t - x^\nu_t\right) &= \left(p_t x'_t - p_t b\right) - \left(p_t x^\nu_t - p_t b\right) \\ &= p_t A x'_{t+1} - p_t A x^\nu_{t+1} + p_t \delta'_{t+1} - p_t \delta'_t \text{ for each } t = 1, 2, \ldots, T-1; \\ p_T\left(x'_T - x^\nu_T\right) &= \left(p_T x'_T - p_T b\right) - \left(p_T x^\nu_T - p_T b\right) = p_T \omega'_{T+1} - p_t \omega^\nu_{T+1} - p_T \delta'_T = -p_T \delta'_T. \end{aligned}$$

Then,

$$\begin{aligned} & w_0 L \sum_{t=1}^{T} \left(x'_t - x^\nu_t\right) = \sum_{t=1}^{T} \left[\frac{p_t - (1+r_t)\, p_{t-1} A}{\prod_{\tau=1}^{t}(1+r_\tau)} \left(x'_t - x^\nu_t\right)\right] \\ & = -p_0 A\left(x'_1 - x^\nu_1\right) + \sum_{t=1}^{T-1} \left[\frac{p_t\left(x'_t - x^\nu_t\right) - p_t A\left(x'_{t+1} - x^\nu_{t+1}\right)}{\prod_{\tau=1}^{t}(1+r_\tau)}\right] + \frac{p_T\left(x'_T - x^\nu_T\right)}{\prod_{t=1}^{T}(1+r_t)} \\ & = \sum_{t=1}^{T} \left[\frac{(1+r_t)\, p_{t-1} - p_t}{\prod_{\tau=1}^{t}(1+r_\tau)} \delta'_t\right] \geqq 0 \qquad (9) \end{aligned}$$

holds for any $\{\delta'_t\}_{t=1}^{T} \subset \mathbb{R}^n_+$, as $p_0 \geqq p_0 A + w_0 L$ yields $(1+r_t)\, p_{t-1} - p_t \geqq \mathbf{0}$ by Proposition 3(i). Therefore, $\{(x^\nu_t, \mathbf{0}, A x^\nu_t)\}_{t=1}^{T}$ constitutes a solution to (MP^ν_T). □

Proof of Corollary 2:

Proof. Focusing on (9), we obtain that the possible path $\{(x^\nu_t, \mathbf{0}, A x^\nu_t)\}_{t=1}^{\infty}$ meets $\liminf_{T\to\infty} L \sum_{t=1}^{T} \left(x'_t - x^\nu_t\right) \geqq 0$ for any $\{x'_t\}_{t=1}^{\infty} \in \mathbf{x}\left(\omega^\nu_1\right)$. □

Proof of Proposition 5:

Proof. (i): Defining $\bar{r} \equiv \limsup_{t\to\infty} r_t$, we let $\bar{r} < R$. We notice that for any $\varepsilon > 0$, there exists t_0 such that $1 + r_t < 1 + \bar{r} + \varepsilon$ for all $t \geqq t_0$. Then, $\frac{\prod_{\tau=t_0+1}^{t}(1+r_\tau)}{(1+\bar{r}+\varepsilon)^{t-t_0}}$ is decreasing with respect to t and converges to 0 as $t \to \infty$, so that for a sufficiently large $T_0 \geqq t_0$,

$$\frac{\prod_{\tau=t_0+1}^{T_0}(1+r_\tau)}{(1+\bar{r}+\varepsilon)^{T_0-t_0}} < \frac{(1+\bar{r}+\varepsilon)^{t_0}}{\prod_{\tau=1}^{t_0}(1+r_\tau)} \;\Leftrightarrow\; \prod_{\tau=1}^{T_0}(1+r_\tau) < (1+\bar{r}+\varepsilon)^{T_0},$$

which yields $\prod_{\tau=1}^{t}(1+r_\tau) < (1+\bar{r}+\varepsilon)^t$ for all $t \geqq T_0$. Thus, for any $\varepsilon \in (0, R - \bar{r})$, we can take T_0 such that $t \geqq T_0$ implies $\prod_{\tau=1}^{t}(1+r_\tau)\, A^{t-1} <$

$(1+\overline{r}+\varepsilon)^t A^{t-1}$. By the choice of $\varepsilon \in (0, R-\overline{r})$, $\prod_{\tau=1}^{t}(1+r_\tau) A^{t-1} < (1+\overline{r}+\varepsilon)^t A^{t-1} \to \mathbf{0}$ as $t \to \infty$ (e.g., Meyer, 2000, 7.10.5). Recall (7) in the proof of Proposition 3. Then, it follows that the right-hand side of (7) converges to the zero vector, which implies $\lim_{t\to\infty}[(1+r_t)p_{t-1}-p_t] = \mathbf{0}$.

(ii): Defining $\underline{r} \equiv \liminf_{t\to\infty} r_t$, we let $\underline{r} > R$. By an argument similar to the above, there exists T_0 such that $t \geqq T_0$ implies $\prod_{\tau=1}^{t}(1+r_\tau) > (1+\underline{r}-\varepsilon)^t$ for any $\varepsilon > 0$. Then, for each i,

$$\begin{aligned}
&\liminf_{t\to\infty}\left[\prod_{\tau=1}^{t}(1+r_\tau)(p_0 - p_0A - w_0L)A^{t-1}\right]_i \\
&\geqq \sup_{\varepsilon>0}\liminf_{t\to\infty}\left[(1+\underline{r}-\varepsilon)^t(p_0-p_0A-w_0L)A^{t-1}\right]_i \\
&= \liminf_{t\to\infty}\left[(1+\underline{r})^t(p_0-p_0A-w_0L)A^{t-1}\right]_i \\
&= (1+\underline{r})(p_0-p_0A-w_0L)\liminf_{t\to\infty}\left\{[(1+\underline{r})A]^{t-1}\right\}_i .
\end{aligned}$$

We know from $\underline{r} > R$ that each entry of $[(1+\underline{r})A]^t$ diverges as $t \to \infty$ (e.g., Meyer, 2000, 7.10.33), so that the rightmost side of the above inequality diverges. Recalling (7) again, we then have $\lim_{t\to\infty}[(1+r_t)p_{t-1}-p_t]_i = \infty$ for all $i = 1, \ldots, n$. This completes the proof. □

6 References

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7 Addendum

7.1 Efficiency

This subsection shows the welfare properties of RSs. In particular, any equilibrium allocation associated with an RS (an *RS-allocation*) starting from any initial aggregate endowments of material input goods, $\overline{\omega}$, which may differ from the minimally necessary capital stock, ω^{Nb}, is shown to be Pareto efficient, and moreover the associated path of aggregate capital accumulation $\{\omega_{t+1}\}_{t=1}^{\infty}$ converges to ω^{Nb} even though $\overline{\omega} \neq \omega^{Nb}$.

To simplify the notation, we denote the elements of aggregate plans by $x_t \equiv \sum_{\nu \in \mathcal{N}} x_t^{\nu}$, $\delta_t \equiv \sum_{\nu \in \mathcal{N}} \delta_t^{\nu}$, and $\omega_{t+1} \equiv \sum_{\nu \in \mathcal{N}} \omega_{t+1}^{\nu}$ for all $t = 1, 2, \ldots$. Then,

Definition A.1: For an economy, $\langle \mathcal{N}, (A, L), u, \overline{\omega} \rangle$, an allocation $\left\{ \left((x_t^{\nu})_{\nu \in \mathcal{N}}, \delta_t, \omega_{t+1} \right) \right\}_{t=1}^{\infty}$ is *feasible* (*from* $\overline{\omega}$) if it satisfies, for each $t = 1, 2, \ldots$, that

(a) $Lx_t^{\nu} \in [0, 1]$ for every $\nu \in \mathcal{N}$;

(b) $(I - A) x_t + \omega_t \geqq Nb + \omega_{t+1}$; and

(c) $Ax_t + \delta_t = \omega_t$.

Definition A.2: For an economy, $\langle \mathcal{N}, (A, L), u, \overline{\omega} \rangle$, a feasible allocation $\left\{ \left((x_t^{\nu})_{\nu \in \mathcal{N}}, \delta_t, \omega_{t+1} \right) \right\}_{t=1}^{\infty}$ is *Pareto efficient* (*in terms of the catching-up criterion from* $\overline{\omega}$) if any feasible allocation $\left\{ \left((x_t^{\prime\nu})_{\nu \in \mathcal{N}}, \delta_t^{\prime}, \omega_{t+1}^{\prime} \right) \right\}_{t=1}^{\infty}$ satisfies the following:

$$\liminf_{T \to \infty} \sum_{t=1}^{T} (Lx_t^{\prime} - Lx_t) \geqq 0.$$

For convenience, let

$$\Gamma(\omega_t) \equiv \left\{ (x_t, \delta_t, \omega_{t+1}) \in \mathbb{R}_+^{3n} \mid \text{(a), (b), \& (c) of Definition A.1} \right\},$$

which is closed in the product topology. The following proposition provides the characterization of all Pareto efficient allocations under $\overline{\omega} = \omega^{Nb}$:

Proposition A.1: Given an economy $\langle \mathcal{N}, (A, L), u, \overline{\omega} \rangle$, let $\overline{\omega} = \omega^{Nb}$. Then, a feasible allocation $\left\{ \left((x_t^{\nu})_{\nu \in \mathcal{N}}, \delta_t, \omega_{t+1} \right) \right\}_{t=1}^{\infty}$ is Pareto efficient if and only if it satisfies $Lx_t = Nvb$ for all $t = 1, 2, \ldots$.

Proof. Let $\left\{\left((x_t^\nu)_{\nu\in\mathcal{N}}, \delta_t, \omega_{t+1}\right)\right\}_{t=1}^\infty$ be feasible for $\langle\mathcal{N}, (A, L), u, \overline{\omega}\rangle$ with $\overline{\omega} = \omega^{Nb}$. Under $\overline{\omega} = \omega^{Nb}$, we can apply the argument in the proof of Proposition 4 to show that any allocation $\left\{\left((x_t^\nu)_{\nu\in\mathcal{N}}, \delta_t, \omega_{t+1}\right)\right\}_{t=1}^\infty$ is feasible in the sense of the conditions (a), (b), and (c) of Definition A.1 if and only if $x_t = (I-A)^{-1}(Nb)$ with $Lx_t^\nu \in [0,1]$ $(\nu \in \mathcal{N})$; $\delta_t = \mathbf{0}$; and $\omega_{t+1} = \omega^{Nb}$ for all $t = 1, 2, \ldots$. This is because the conditions of Definition A.1 are identical to the conditions for RS allocations, except for the catching up optimality condition. Note that, by $L(I-A)^{-1}(Nb) < N$, there exists a profile $(x_t^\nu)_{\nu\in\mathcal{N}}$ such that $Lx_t^\nu \in [0,1]$ $(\nu \in \mathcal{N})$ for all $t = 1, 2, \ldots$.

Thus, by such a characterization, it follows that for any feasible allocation $\left\{\left((x_t^\nu)_{\nu\in\mathcal{N}}, \delta_t, \omega_{t+1}\right)\right\}_{t=1}^\infty$, $Lx_t = Nvb$ holds for all $t = 1, 2, \ldots$, which implies that any feasible allocation is Pareto efficient in terms of the catching-up criterion. □

To consider the characterization of all Pareto efficient allocations under the case of $\overline{\omega} \geq \omega^{Nb}$ and $\overline{\omega} \leq \omega^{Nb} + Nb$, we prove the following two lemmas:

Lemma I: Given an economy $\langle\mathcal{N}, (A, L), u, \overline{\omega}\rangle$ with $\omega^{Nb} \leq \overline{\omega} \leq \omega^{Nb} + Nb$, let $\left\{\left((x_t^\nu)_{\nu\in\mathcal{N}}, \delta_t, \omega_{t+1}\right)\right\}_{t=1}^\infty$ be a feasible allocation. Suppose that there exists $T < \infty$ such that $\omega_T = \omega^{Nb}$. If $\left\{\left((x_t^\nu)_{\nu\in\mathcal{N}}, \delta_t, \omega_{t+1}\right)\right\}_{t=1}^\infty$ is Pareto efficient, then it satisfies $\sum_{t=1}^{T-1} Lx_t = TNvb - v\left(\overline{\omega} - \omega^{Nb}\right)$ and $Lx_t = Nvb$ for all $t \geqq R$.

Proof. Let $\omega^{Nb} \leq \overline{\omega} \leq \omega^{Nb} + Nb$ and let $\left\{\left((x_t^\nu)_{\nu\in\mathcal{N}}, \delta_t, \omega_{t+1}\right)\right\}_{t=1}^\infty$ be a feasible path. Note that $x_1 = \mathbf{0}$ cannot be feasible by $\overline{\omega} \ngeq \omega^{Nb} + Nb$, and therefore $Lx_1 > 0$ holds. Suppose that there exists $T < \infty$ such that $\omega_T = \omega^{Nb}$. Then, $x_t = (I-A)^{-1}(Nb)$; $\delta_t = \mathbf{0}$; and $\omega_{t+1} = \omega^{Nb}$ must hold for all $t \geqq T$, by the same argument as the proof of Proposition A.1.

Fix $T < \infty$ and consider the following program:

$$\min_{\{(x_t, \delta_t, \omega_{t+1})\}_{t=1}^T} \sum_{t=1}^T Lx_t, \text{ s.t. } \begin{array}{l} \forall t = 1, 2, \ldots, T,\ Ax_t + \delta_t = \omega_t; \\ Lx_t \leqq N;\ (I-A)x_t + \omega_t \geqq Nb + \omega_{t+1}; \\ \omega_{T+1} = \omega^{Nb}; \quad \omega_1 \equiv \overline{\omega} \text{ given.} \end{array} \tag{MP_T}$$

Let $\left\{\left(x_t^*, \delta_t^*, \omega_{t+1}^*\right)\right\}_{t=1}^T$ be the solution to the program (MP_T) and define a allocation $\left\{\left(x_t^*, \delta_t^*, \omega_{t+1}^*\right)\right\}_{t=1}^\infty$ such that $x_t^* \equiv (I-A)^{-1}(Nb)$; $\delta_t^* \equiv \mathbf{0}$; and

$\omega_{t+1} \equiv \omega^{Nb}$ for $t = T+1, T+2, \ldots$. Note that $\left\{ \left(x_t^*, \delta_t^*, \omega_{t+1}^* \right) \right\}_{t=1}^{\infty}$ satisfies the conditions (b) and (c) of Definition A.1. Then, $\left\{ \left(\left(x_t^{*\nu} \right)_{\nu \in \mathcal{N}}, \delta_t^*, \omega_{t+1}^* \right) \right\}_{t=1}^{\infty}$ with $\sum_{\nu \in \mathcal{N}} x_t^{*\nu} = x_t^*$ $(t = 1, 2, \ldots)$ is feasible if and only if for every $\nu \in \mathcal{N}$, $Lx_t^{*\nu} \in [0,1]$ for all $t = 1, 2, \ldots$. Now, we can prove that if a feasible path $\left\{ \left(\left(x_t^{\nu} \right)_{\nu \in \mathcal{N}}, \delta_t, \omega_{t+1} \right) \right\}_{t=1}^{\infty}$ with $\omega_T = \omega^{Nb}$ is Pareto efficient, then $\left\{ \left(\left(\sum_{\nu \in \mathcal{N}} x_t^{\nu} \right), \delta_t, \omega_{t+1} \right) \right\}_{t=1}^{\infty}$ with $\omega_{T+1} = \omega^{Nb}$ is a solution to (MP_T).

To this end, suppose not: a Pareto efficient path $\left\{ \left(\left(x_t^{\nu} \right)_{\nu \in \mathcal{N}}, \delta_t, \omega_{t+1} \right) \right\}_{t=1}^{\infty}$, with $\omega_{T+1} = \omega^{Nb}$, does not constitutes a solution to (MP_T). Note that because of feasibility, it satisfies all the constraints in (MP_T) for any $t \leqq T$ and $x_t = (I - A)^{-1}(Nb)$; $\delta_t = \mathbf{0}$; and $\omega_{t+1} = \omega^{Nb}$ for any $t > T$, as we have seen. Since $\{(x_t, \delta_t, \omega_{t+1})\}_{t=1}^{T}$ cannot be the solution to (MP_T) under $\omega_{T+1} = \omega^{Nb}$, we have $\sum_{t=1}^{T} Lx_t > \sum_{t=1}^{T-1} Lx_t^*$. Then,

$$\liminf_{T \to \infty} \sum_{t=1}^{T} \left(Lx_t^* - Lx_t \right) = \sum_{t=1}^{T} Lx_t^* - \sum_{t=1}^{T} Lx_t < 0,$$

which contradicts the Pareto efficiency of $\left\{ \left(\left(x_t^{\nu} \right)_{\nu \in \mathcal{N}}, \delta_t, \omega_{t+1} \right) \right\}_{t=1}^{\infty}$.

Finally, we demonstrate that $\left\{ \left(\left(x_t^{\nu} \right)_{\nu \in \mathcal{N}}, \delta_t, \omega_{t+1} \right) \right\}_{t=1}^{\infty}$ has the above form of $\left\{ \left(\left(x_t^{*\nu} \right)_{\nu \in \mathcal{N}}, \delta_t^*, \omega_{t+1}^* \right) \right\}_{t=1}^{\infty}$ if and only if it satisfies $\sum_{t=1}^{T} Lx_t = TNvb - v\left(\overline{\omega} - \omega^{Nb} \right)$ and $Lx_t = Nvb$ for all $t > T$. By its construction, $\left\{ \left(\left(x_t^{\nu} \right)_{\nu \in \mathcal{N}}, \delta_t, \omega_{t+1} \right) \right\}_{t=1}^{\infty}$ satisfies $Lx_t = Nvb$ for all $t > T$, and conversely, $Lx_t \neq Nvb$ for some $t > T$ violates the feasibility of $\left\{ \left(\left(x_t^{\nu} \right)_{\nu \in \mathcal{N}}, \delta_t, \omega_{t+1} \right) \right\}_{t=1}^{\infty}$. Moreover, since $\{(x_t, \delta_t, \omega_{t+1})\}_{t=1}^{T}$ is the solution to (MP_T), it satisfies the constraints of (MP_T) with equality under the minimization, i.e., $(I - A)\,x_t + \omega_t = Nb + \omega_{t+1}$ for any $t \leqq T$. We have, from the summation with respect to t, that $\sum_{t=1}^{T} x_t = T(I - A)^{-1}(Nb) - (I - A)^{-1}\left(\overline{\omega} - \omega^{Nb} \right)$. Then, the solution to (MP_T), $\{(x_t, \delta_t, \omega_{t+1})\}_{t=1}^{T}$ satisfies $\sum_{t=1}^{T} Lx_t = TNvb - v\left(\overline{\omega} - \omega^{Nb} \right)$. This completes the proof. □

Lemma II: Given an economy $\langle \mathcal{N}, (A, L), u, \overline{\omega} \rangle$ with $\omega^{Nb} \leq \overline{\omega} \leq \omega^{Nb} + Nb$, let $\left\{ \left(\left(x_t^{\nu} \right)_{\nu \in \mathcal{N}}, \delta_t, \omega_{t+1} \right) \right\}_{t=1}^{\infty}$ be a Pareto efficient allocation. Then, it satisfies $\sum_{t=1}^{\infty} \left(Lx_t - Nvb \right) = -v\left(\overline{\omega} - \omega^{Nb} \right)$ and $\lim_{t \to \infty} Lx_t = Nvb$ if and only if it fulfills $\lim_{t \to \infty} \omega_t = \omega^{Nb}$.

Proof. Let $\omega^{Nb} \leq \overline{\omega} \leq \omega^{Nb} + Nb$. Note first that a Pareto efficient path $\left\{ \left(\left(x_t^{\nu} \right)_{\nu \in \mathcal{N}}, \delta_t, \omega_{t+1} \right) \right\}_{t=1}^{\infty}$ satisfies the condition (b) of Definition A.1 with equality for all $t = 1, 2, \ldots$. Suppose, contrarily, that there exists a Pareto efficient path $\left\{\left((x_t^\nu)_{\nu \in \mathcal{N}}, \delta_t, \omega_{t+1}\right)\right\}_{t=1}^{\infty}$ such that $(I - A)\, x_T + \omega_T \geq Nb + \omega_{T+1}$ for some $T < \infty$. Then, taking $\varepsilon_T \geq \mathbf{0}$ with $(I - A)\, x_T - \varepsilon_T + \omega_T = Nb + \omega_{T+1}$ and setting $x'_T \equiv x_T - (I - A)^{-1} \varepsilon_T$, we can construct a feasible path $\left\{\left((x_t'^\nu)_{\nu \in \mathcal{N}}, \delta'_t, \omega'_{t+1}\right)\right\}_{t=1}^{\infty}$ such that $x'_t \equiv x_t$ for $t = 1, 2, \ldots, T-1, T+1, \ldots$. Since $x'_T < x_T$ by $\varepsilon_T \geq \mathbf{0}$ and $(I - A)^{-1} > \mathbf{0}$, we have, from $L > \mathbf{0}$, that $Lx'_T < Lx_T$ and

$$\liminf_{T \to \infty} \sum_{t=1}^{T} (Lx'_t - Lx_t) = (Lx'_T - Lx_T) < 0,$$

which contradicts the Pareto efficiency of $\left\{\left((x_t^\nu)_{\nu \in \mathcal{N}}, \delta_t, \omega_{t+1}\right)\right\}_{t=1}^{\infty}$. This implies that any Pareto efficient path satisfies condition (b) of Definition A.1 with equality for all $t = 1, 2, \ldots$. Thus, taking $\Delta_t \equiv \omega_t - \omega^{Nb} \geqq \mathbf{0}$ ($t = 1, 2, \ldots$), we can denote any Pareto efficient path $\left\{\left((x_t^\nu)_{\nu \in \mathcal{N}}, \delta_t, \omega_{t+1}\right)\right\}_{t=1}^{\infty}$ by $x_t = (I - A)^{-1} (Nb - \Delta_t + \Delta_{t+1})$ and $\delta_t = (I - A)^{-1} (\Delta_t - A\Delta_{t+1})$ for $t = 1, 2, \ldots$. We notice from $x_1 = (I - A)^{-1} \left(Nb - \overline{\omega} + \omega^{Nb} + \Delta_2\right)$ that for a period $T < \infty$,

$$\sum_{t=1}^{T} Lx_t = TNvb - v\left(\overline{\omega} - \omega^{Nb}\right) + v\Delta_{T+1}.$$

For the "if" part, let $\lim_{t \to \infty} \omega_t = \omega^{Nb}$. Then, we have from $\lim_{t \to \infty} v\Delta_t = 0$ in (A1) that $\sum_{t=1}^{\infty} (Lx_t - Nvb) = -v\left(\overline{\omega} - \omega^{Nb}\right)$. Since $\Gamma(\overline{\omega})$ is closed in the product topology, $\lim_{t \to \infty} \left(x'_t, \delta'_t, \omega'_{t+1}\right) \in \Gamma(\overline{\omega})$ with $\lim_{t \to \infty} \omega'_t = \omega^{Nb}$ implies $\lim_{t \to \infty} x'_t = (I - A)^{-1} (Nb)$, and therefore, $\lim_{t \to \infty} Lx_t = Nvb$. To show the "only if" part, we suppose conversely that it violates $\lim_{t \to \infty} \omega_t = \omega^{Nb}$, i.e., $\limsup_{t \to \infty} \omega_{it} > \omega_i^{Nb}$ for some i, where ω_{it} and ω_i^{Nb} are respectively the i-th element of ω_t and ω^{Nb}. Then, we have from (A1) and $v > \mathbf{0}$ that

$$\limsup_{T \to \infty} \sum_{t=1}^{T} (Lx_t - Nvb) = -v\left(\overline{\omega} - \omega^{Nb}\right) + \limsup_{T \to \infty} v\Delta_T > -v\left(\overline{\omega} - \omega^{Nb}\right),$$

which contradicts $\sum_{t=1}^{\infty} (Lx_t - Nvb) = -v\left(\overline{\omega} - \omega^{Nb}\right)$. This completes the proof. □

Proposition A.2: Given an economy $\langle \mathcal{N}, (A, L), u, \overline{\omega} \rangle$, let $\omega^{Nb} \leq \overline{\omega} \leq \omega^{Nb} + Nb$. Then, a feasible allocation $\left\{\left((x_t^\nu)_{\nu \in \mathcal{N}}, \delta_t, \omega_{t+1}\right)\right\}_{t=1}^{\infty}$ is Pareto efficient if and only if it satisfies $\sum_{t=1}^{\infty}(Lx_t - Nvb) = -v\left(\overline{\omega} - \omega^{Nb}\right)$ and $\lim_{t\to\infty} Lx_t = Nvb$.

Proof. Let $\omega^{Nb} \leq \overline{\omega} \leq \omega^{Nb} + Nb$. We first show the "only if" part. To see this, suppose, contrarily, that a Pareto efficient path $\left\{\left((x_t^{\nu})_{\nu\in\mathcal{N}}, \delta_t, \omega_{t+1}\right)\right\}_{t=1}^{\infty}$ violates $\sum_{t=1}^{\infty}(Lx_t - Nvb) = -v\left(\overline{\omega} - \omega^{Nb}\right)$ or $\lim_{t\to\infty} Lx_t = Nvb$. By Lemma II, this is equivalent to the violation of $\lim_{t\to\infty}\omega_t = \omega^{Nb}$, i.e., $\limsup_{t\to\infty}\omega_{it} > \omega_i^{Nb}$ for some commodity i. As we have seen in the proof of Lemma II, $\left\{\left((x_t^{\nu})_{\nu\in\mathcal{N}}, \delta_t, \omega_{t+1}\right)\right\}_{t=1}^{\infty}$ satisfies (b) of Definition A.1 with equality for all $t = 1, 2, \ldots$ by its Pareto efficiency. Then, taking $\Delta_t \equiv \omega_t - \omega^{Nb} \geq \mathbf{0}$ $(t = 1, 2, \ldots)$, we have (A1). Let $x_1^* \equiv (I - A)^{-1}\left(Nb - \overline{\omega} + \omega^{Nb}\right)$ and $\delta_1^* \equiv \overline{\omega} - Ax_1^* = (I - A)^{-1}\left(\overline{\omega} - \omega^{Nb}\right)$. Define a path $\left\{\left(x_t^*, \delta_t^*, \omega_{t+1}^*\right)\right\}_{t=1}^{\infty}$ such that $x_t^* \equiv (I - A)^{-1}(Nb)$; $\delta_t^* \equiv \mathbf{0}$; and $\omega_{t+1} \equiv \omega^{Nb}$ for $t = 2, 3, \ldots$. As in the proof of Lemma I, it is feasible if and only if for every $\nu \in \mathcal{N}$, $Lx_t^{*\nu} \in [0, 1]$ for all $t = 1, 2, \ldots$. For a period $T < \infty$, we have from the summation with respect to t that

$$\sum_{t=1}^{T} Lx_t^* = \sum_{t=1}^{T} Nvb - v\left(\overline{\omega} - \omega^{Nb}\right).$$

Then, we obtain from $\Delta_t \geq \mathbf{0}$ for all $t = 1, 2, \ldots$, that

$$\liminf_{T\to\infty}\sum_{t=1}^{T}\left(Lx_t^* - Lx_t\right) = \liminf_{T\to\infty}\left(-v\Delta_{T+1}\right) < 0.$$

Note that since $\limsup_{t\to\infty}\omega_{it} \geq \omega_i^{Nb}$ for some i, this inequality holds. This contradicts the Pareto efficiency of $\left\{\left((x_t^{\nu})_{\nu\in\mathcal{N}}, \delta_t, \omega_{t+1}\right)\right\}_{t=1}^{\infty}$, which completes the proof of the "only if" part.

To show the "if" part, we will consider all admissible feasible paths. Note that since a feasible path $\left\{\left((x_t^{\nu})_{\nu\in\mathcal{N}}, \delta_t, \omega_{t+1}\right)\right\}_{t=1}^{\infty}$ satisfies $\omega_t \geqq \omega^{Nb}$ for all $t = 2, 3, \ldots$, it exactly fulfills one of the following two cases: either (I) $\omega_t \geq \omega^{Nb}$ holds for all $t = 2, 3, \ldots$; or (II) there exists $\tau < \infty$ such that $\omega_t \geq \omega^{Nb}$ for any $t \leqq \tau$ and $\omega_t = \omega^{Nb}$ for any $t > \tau$. For the case of (I), let $\left\{\left((x_t'^{\nu})_{\nu\in\mathcal{N}}, \delta_t', \omega_{t+1}'\right)\right\}_{t=1}^{\infty}$ be a feasible path such that $\sum_{t=1}^{\infty}(Lx_t' - Nvb) = -v\left(\overline{\omega} - \omega^{Nb}\right)$ and $\lim_{t\to\infty} Lx_t' = Nvb$. For the case of (II), note that, any Pareto efficient path satisfies $\sum_{t=1}^{\tau} Lx_t = \tau Nvb - v\left(\overline{\omega} - \omega^{Nb}\right)$ and $Lx_t = Nvb$ for all $t > \tau$, by Lemma I. Let $\left\{\left((x_t''^{\nu})_{\nu\in\mathcal{N}}, \delta_t'', \omega_{t+1}''\right)\right\}_{t=1}^{\infty}$ satisfy

$\sum_{t=1}^{\tau} Lx''_t = \tau Nvb - v\left(\overline{\omega} - \omega^{Nb}\right)$ and $Lx''_t = Nvb$ $(t > \tau)$ in case (II). Then, it suffices to check that $\left\{\left((x'^{\nu}_t)_{\nu\in\mathcal{N}}, \delta'_t, \omega'_{t+1}\right)\right\}_{t=1}^{\infty}$ and $\left\{\left((x''^{\nu}_t)_{\nu\in\mathcal{N}}, \delta''_t, \omega''_{t+1}\right)\right\}_{t=1}^{\infty}$ are equivalent in terms of the catching-up optimality. By a calculation similar to that above, we have, for any $T > \tau$,

$$\sum_{t=1}^{T} (Lx_t - Lx'_t) = -v\Delta'_{T+1}.$$

Then, we obtain from $\lim_{t\to\infty} \Delta'_t = \mathbf{0}$ that

$$\liminf_{T\to\infty} \sum_{t=1}^{T} (Lx_t - Lx'_t) = \liminf_{T\to\infty} \left(-v\Delta'_{T+1}\right) = -\lim_{T\to\infty} v\Delta'_{T+1} = 0,$$

which completes the proof. □

Using the characterizations of Pareto efficient allocations, we now show the first fundamental theorem of welfare economics in terms of RSs.

Proposition A.3: Given an economy $\left\langle \mathcal{N}, (A, L), u, (\omega_1^{\nu})_{\nu\in\mathcal{N}} \right\rangle$ with $\omega^{Nb} \leqq \overline{\omega} \leq \omega^{Nb} + Nb$, let $\left\{ \left(x_t^{*\nu}, \delta_t^{*\nu}, \omega_{t+1}^{*\nu}\right)_{\nu\in\mathcal{N}} \right\}_{t=1}^{\infty}$ be an RS allocation associated with $\left\{p^*_{t-1}; (w_t^{\nu*}, r_t^{\nu*})_{\nu\in\mathcal{N}}\right\}_{t=1}^{\infty}$. Then, it is Pareto efficient.

Proof. Let $\left\{ \left(x_t^{*\nu}, \delta_t^{*\nu}, \omega_{t+1}^{*\nu}\right)_{\nu\in\mathcal{N}} \right\}_{t=1}^{\infty}$ be an RS allocation. We firstly prove that $x_t^* > \mathbf{0}$ for all $t = 1, 2, \ldots$. To show this, we fix t and, suppose on the contrary that $x_{it}^* = 0$ for some commodity i. Note that, for any t, $x_t^* = \mathbf{0}$ cannot be an RS activity by $\overline{\omega} \ngeqq \omega^{Nb} + Nb$. Since $x_t^* = (I - A)^{-1}\left(Nb + \omega_{t+1}^* - \omega_t^* + \varepsilon\right)$ for a suitable $\varepsilon \in \mathbb{R}^n_+$ by (b) of Definition 1, the supposition of $x_{it}^* = 0$ implies $Nb + \omega_{t+1}^* - \omega_t^* + \varepsilon = \mathbf{0}$ by $(I - A)^{-1} > \mathbf{0}$. This contradicts to $x_t^* \geq \mathbf{0}$. Then, $x_t^* > \mathbf{0}$ for all $t = 1, 2, \ldots$. Fixing t arbitrarily, we have that for each commodity i, there exists $\nu \in \mathcal{N}$ such that $x_{it}^{*\nu} > 0$. We then notice that $p_{it}^* = (1 + r_t^{\nu*})\, p_{t-1}^* A e_i + w_t^{\nu*} L_i$ for i satisfying $x_{it}^{*\nu} > 0$, where e_i is the i-th unit vector. Indeed, $p_{it}^* < (1 + r_t^{\nu*})\, p_{t-1}^* A e_i + w_t^{\nu*} L_i$ $(\forall i)$ or $p_{it}^* > (1 + r_t^{\nu*})\, p_{t-1}^* A e_i + w_t^{\nu*} L_i$ $(\forall i)$ necessarily violates constraint (4). Moreover, if $p_{it}^* < (1 + r_t^{\nu*})\, p_{t-1}^* A e_i + w_t^{\nu*} L_i$ for some i, then fulfilling (4) implies $p_{jt}^* > (1 + r_t^{\nu*})\, p_{t-1}^* A e_j + w_t^{\nu*} L_j$ for some j satisfying $x_{jt}^{*\nu} > 0$. Taking

$x_t^{\prime\nu} \leq x_t^{*\nu}$ such that $x_{it}^{\prime\nu} < x_{it}^{*\nu}$ and $x_{jt}^{\prime\nu} < x_{jt}^{*\nu}$ to meet (4), and choosing a suitable $\delta_t^{\prime\nu}$, we can construct a possible allocation at t, $\left(x_t^{\prime\nu}, \delta_t^{\prime\nu}, \omega_{t+1}^{*\nu}\right)$, which is a contradiction to ν's optimality. We thus have $p_{it}^* = (1 + r_t^{\nu *})\, p_{t-1}^* A e_i + w_t^{\nu *} L_i$ for i satisfying $x_{it}^{*\nu} > 0$.

We now suppose contrarily that it is not Pareto efficient. This is equivalent to $\limsup_{t\to\infty} \omega_{it}^* > \omega_i^{Nb}$ for some commodity i, by Lemma II and Proposition A.2. Note that, for each $t = 2, 3, \ldots$, we can take $i\,(t) \in \left\{i \mid \omega_{it}^* - \omega_i^{Nb} \geqq \omega_{jt}^* - \omega_j^{Nb}, \forall j\right\}$. Setting $\Delta_{i(t)}^{\prime} \equiv \max_i \left\{\omega_{it}^* - \omega_i^{Nb} \mid i = 1, 2, \ldots, n\right\} > 0$ for each $t = 2, 3, \ldots$, we have $\liminf_{t\to\infty} \Delta_{i(t)}^{\prime} > 0$, which implies $\inf_t \Delta_{i(t)}^{\prime} > 0$. (Indeed, if $\inf_t \Delta_{i(t)}^{\prime} = 0$, then $\Delta_{i(t)}^{\prime} = 0$ for some t, which is a contradiction.) Taking $\varepsilon_{i(1)} \in \left(0, \frac{\min_i v_i}{\max_i v_i} \inf_t \Delta_{i(t)}^{\prime}\right)$ and inductively defining $\varepsilon_{i(t+1)} \equiv \frac{v_{i(t)}}{v_{i(t+1)}} \varepsilon_{i(t)}$ for $t = 1, 2, \ldots$, we have that for any t,

$$
\begin{aligned}
\varepsilon_{i(t+1)} &= \frac{v_{i(t)}}{v_{i(t+1)}} \varepsilon_{i(t)} = \frac{v_{i(t)}}{v_{i(t+1)}} \frac{v_{i(t-1)}}{v_{i(t)}} \varepsilon_{i(t-1)} \\
&= \cdots = \frac{v_{i(1)}}{v_{i(t+1)}} \varepsilon_{i(1)} \\
&< \frac{v_{i(1)}}{v_{i(t+1)}} \frac{\min_i v_i}{\max_i v_i} \inf_t \Delta_{i(t)}^{\prime} \\
&\leqq \inf_t \Delta_{i(t)}^{\prime}.
\end{aligned}
$$

Then, we obtain $\omega_{i(t)}^* - \varepsilon_{i(t)} > \omega_{i(t)}^{Nb}$ and $v_{i(t)} \varepsilon_{i(t)} = v_{i(t+1)} \varepsilon_{i(t+1)}$ for all $t = 2, 3, \ldots$.

Setting $\varepsilon_t \equiv \varepsilon_{i(t)t} e_{i(t)}$ $(t = 2, 3, \ldots)$, and taking $\omega_t^{\prime} \equiv \omega_t^* - \varepsilon_t$ $(t = 2, 3, \ldots)$; $x_1^{\prime} \equiv (I - A)^{-1} (Nb - \overline{\omega} + \omega_2^{\prime})$; $x_t^{\prime} \equiv (I - A)^{-1} \left(Nb - \omega_t^{\prime} + \omega_{t+1}^{\prime}\right)$ $(t = 2, 3, \ldots)$; $\delta_1^{\prime} \equiv \overline{\omega} - A x_1^{\prime}$; and $\delta_t^{\prime} \equiv \omega_t - A x_t^{\prime}$ $(t = 2, 3, \ldots)$, we construct a path $\left\{\left(x_t^{\prime}, \delta_t^{\prime}, \omega_{t+1}^{\prime}\right)\right\}_{t=1}^{\infty}$ satisfying the feasibility conditions (b) and (c) of Definition A.1. Note that it is Pareto superior to $\left\{\left(x_t^{*\nu}, \delta_t^{*\nu}, \omega_{t+1}^{*\nu}\right)_{\nu \in \mathcal{N}}\right\}_{t=1}^{\infty}$, since $x_1^{\prime} = (I - A)^{-1} (Nb - \overline{\omega} + \omega_2^*) - (I - A)^{-1} \varepsilon_2 < x_1^*$ and $L x_t^{\prime} = v \left(Nb - \omega_t^* + \omega_{t+1}^*\right) + v \left(\varepsilon_t - \varepsilon_{t+1}\right) \leqq L x_t^* + \left(v_{i(t)} \varepsilon_{i(t)} - v_{i(t+1)} \varepsilon_{i(t+1)}\right) = L x_t^*$ $(t = 2, 3, \ldots)$ by the construction. Then, for each $t = 1, 2, \ldots$, there exists a suitable assignment $\left(x_t^{\prime\nu}, \delta_t^{\prime\nu}, \omega_{t+1}^{\prime\nu}\right)_{\nu \in \mathcal{N}}$ such that: $x_t^{\prime\nu} \leqq x_t^{*\nu}$ for any $\nu \in \mathcal{N}$ and $x_t^{\prime\mu} \leq x_t^{*\mu}$ for some $\mu \in \mathcal{N}$; $p_0^* \delta_1^{\prime} = p_0^* \omega_1^{\nu} - p_0^* A x^{\prime\nu}$ and $p_1^* \omega_2^{\prime\nu} = p_1^* x_1^{\prime} + p_1^* \delta_1^{\prime\nu} - p_1^* b$; and $p_{t-1}^* \delta_t^{\prime} = p_{t-1}^* \omega_t^{\nu} - p_{t-1}^* A x^{\prime\nu}$ and $p_t^* \omega_{t+1}^{\prime\nu} = p_t^* x_t^{\prime} + p_t^* \delta_t^{\prime\nu} - p_t^* b$ for all $t = 2, 3, \ldots$. This implies that for each $\nu \in \mathcal{N}$, $\left\{\left(x_t^{\prime\nu}, \delta_t^{\prime\nu}, \omega_{t+1}^{\prime\nu}\right)_{\nu \in \mathcal{N}}\right\}_{t=1}^{\infty}$ meets the constraints of (1)–(3). Moreover, since $p^*_{it} = (1 + r^{\nu *}_t)\, p^*_{it-1} A e_i + w^{\nu *}_t L_i$ for $\nu \in \mathcal{N}$ satisfying $x^{*\nu}_{it} > 0$ as noted above, it also meets constraint (4). By the construction, this assignment contradicts the claim that $\{x^{*\nu}_t\}^{\infty}_{t=1}$ is catching-up optimal for every $\nu \in \mathcal{N}$. □

By Lemma II, Propositions A.2 and A.3, any RSs in world economies with the leisure preference should approach to the minimally necessary capital stock in the infinite limit of time, i.e., $\omega_t \to \omega^{Nb}$ as $t \to \infty$.

7.2 Existence

This section proves the existence of an imperfectly specialized IRS for the pre-industrial world economies analyzed in the paper. The proof shows the consistency of the formal framework and establishes that the impossibility result on the persistent UE free trade equilibria characterized in Theorem 2 and Corollary 4 does not hold vacuously.

For the existence of RSs with non-stationary relative prices of commodities, we first provide suitable conditions for ruling out a kind of "factor-intensity reversal," and thereby find tractable price paths.[21] Let q be the left-hand side Perron-Frobenius eigenvector associated with the Perron-Frobenius eigenvalue of A, $\frac{1}{1+R}$. Define $\tilde{p}_0(\varsigma_0) \equiv \varsigma_0 q + w_0 v$ for $\varsigma_0 > 0$.

Lemma III: Take $p_0 = \tilde{p}_0(\varsigma_0)$ for an arbitrary $\varsigma_0 > 0$, and let $\{p_{t-1}; (w_t, r_t)\}^{\infty}_{t=1} = \mathbf{p}(p_0, w_0, \mathbf{r})$ for $\mathbf{r} \subset \overline{\mathbb{R}}_+$. Then, $(1 + r_t)\, p_{t-1} - p_t = \varsigma_0 R \prod^{t}_{\tau=1} \left(\frac{1+r_\tau}{1+R}\right) q$ for all $t = 1, 2, \ldots$.

Proof. Let $p_0 = \varsigma_0 q + w_0 v$. Note that $v = L(I - A)^{-1} = L + L(I - A)^{-1} A = L + vA$. Then, $\{p_{t-1}; (w_t, r_t)\}^{\infty}_{t=1} = \mathbf{p}(p_0, w_0, \mathbf{r})$ yields

$$\begin{aligned} p_1 &= (1 + r_1)(p_0 A + w_0 L) \\ &= (1 + r_1)\left[(\varsigma_0 q + w_0 v) A + w_0 L\right] \\ &= (1 + r_1)\left(\frac{\varsigma_0}{1+R} q + w_0 v\right), \end{aligned}$$

[21] The standard Heckscher-Ohlin model traditionally assumes no factor-intensity reversal for the analysis, in order to ensure factor price equalization.

so that

$$\left(1+r_1\right)p_0-p_1=\left(1+r_1\right)\left[\varsigma_0 q+w_0 v-\left(\frac{\varsigma_0}{1+R}q+w_0 v\right)\right]=\varsigma_0 R\frac{1+r_1}{1+R}q.$$

Moreover, supposing $\left(1+r_t\right)p_{t-1}-p_t=\varsigma_0 R\prod_{\tau=1}^{t}\left(\frac{1+r_\tau}{1+R}\right)q$ for any t, we obtain from (7) in the proof of Proposition 3 that

$$\begin{aligned}\left(1+r_{t+1}\right)p_t-p_{t+1}&=\left(1+r_{t+1}\right)\left[\left(1+r_t\right)p_{t-1}-p_t\right]A\\&=\frac{1+r_{t+1}}{1+R}\left[\left(1+r_t\right)p_{t-1}-p_t\right]\\&=\varsigma_0 R\prod_{\tau=1}^{t+1}\left(\frac{1+r_\tau}{1+R}\right)q.\end{aligned}$$

This completes the proof. □

By Proposition 3-(ii) and Corollary 2, if $L=\varsigma q$ for some $\varsigma>0$, then the price vector $\tilde{p}_0\left(\varsigma_0\right)$ in Lemma III becomes the labor-value pricing and the stationary relative prices of commodities. As our subject here is to examine the existence of RSs with non-stationary relative prices of commodities, we assume $L\neq\varsigma q$ for any $\varsigma>0$.

Note that if $L\neq\varsigma q>0$, then $\frac{vAe_i}{L_i}\neq\frac{vAe_j}{L_j}$ for some i,j. Indeed, otherwise $\frac{vAe_i}{L_i}=\frac{vAe_j}{L_j}$ holds for any (i,j), which is equivalent to $L=\varsigma vA$ for some $\varsigma>0$. Then, by the Neumann series $(I-A)^{-1}=\sum_{\tau=0}^{\infty}A^\tau$,

$$L=\varsigma vA=\varsigma L\left(I-A\right)^{-1}A=\varsigma LA\left(I-A\right)^{-1}\ \Leftrightarrow\ L=\left(1+\varsigma\right)LA,$$

so that $L>\mathbf{0}$ is the eigenvector for A and must be the unique (up to scalar) Perron-Frobenius vector by the indecomposability of A. This contradicts $L\neq\varsigma q$ for any $\varsigma>0$.

We additionally have the following observation.

Lemma IV: Suppose that (A,L) satisfies $\frac{vAe_i}{L_i}\neq\frac{vAe_k}{L_k}$ for some $i,k=1,\ldots,n$. Take $p_0=\tilde{p}_0\left(\varsigma_0\right)$ for $\varsigma_0>0$, and let $\left\{p_{t-1};\left(w_t,r_t\right)\right\}_{t=1}^{\infty}=\mathbf{p}\left(p_0,w_0,\mathbf{r}\right)$ for $\mathbf{r}\subset\overline{\mathbb{R}}_+$. Then, there exists a commodity j^* for any $t=1,2,\ldots$, such that $\frac{p_tAe_i}{p_tAe_{j^*}}\geqq\frac{p_t(I-A)e_i}{p_t(I-A)e_{j^*}}$ for all i and $\frac{p_tAe_{i^*}}{p_tAe_{j^*}}>\frac{p_t(I-A)e_{i^*}}{p_t(I-A)e_{j^*}}$ for some i^*, by setting a sufficiently small $\varsigma_0>0$.

Proof. By Lemma III, taking $p_0 = \varsigma_0 q + w_0 v$ yields $(1+r_t)\, p_{t-1} - p_t = \varsigma_0 R \prod_{\tau=1}^{t} \left(\frac{1+r_\tau}{1+R}\right) q$, so that

$$\begin{aligned} p_t (I-A) &= [(1+r_t)\, p_{t-1} - p_t]\, A + \prod_{\tau=1}^{t} (1+r_\tau)\, w_0 L \\ &= \prod_{\tau=1}^{t} (1+r_\tau) \left[\frac{\varsigma_0 R}{(1+R)^t} qA + w_0 L\right]. \end{aligned} \tag{A3}$$

Moreover, since $\frac{1}{1+R} q = qA$ is equivalent to $q = RqA\,(I-A)^{-1}$,

$$\begin{aligned} p_t A &= \prod_{\tau=1}^{t} (1+r_\tau) \left[\frac{\varsigma_0 R}{(1+R)^t} qA\,(I-A)^{-1} + v\right] A \\ &= \prod_{\tau=1}^{t} (1+r_\tau) \left[\frac{\varsigma_0}{(1+R)^t} qA + vA\right]. \end{aligned} \tag{A4}$$

Define $J \equiv \left\{ j \mid \forall i,\ \frac{vAe_i}{L_i} \geqq \frac{vAe_j}{L_j} \right\}$, and take (i, j) for $i \notin J$ and $j \in J$. Then, $vAe_i L_j > vAe_j L_i$. We have from (A3) and (A4) that

$$\begin{aligned} &\frac{p_1 Ae_i}{p_1 Ae_j} - \frac{p_1 (I-A)\, e_i}{p_1 (I-A)\, e_j} \\ &= \frac{\frac{\varsigma_0}{1+R} qAe_i + vAe_i}{\frac{\varsigma_0}{1+R} qAe_j + vAe_j} - \frac{\frac{\varsigma_0 R}{1+R} qAe_i + L_i}{\frac{\varsigma_0 R}{1+R} qAe_j + L_j} \\ &= \frac{\frac{\varsigma_0}{1+R} qA\,[(L_j - RvAe_j)\, e_i - (L_i - RvAe_i)\, e_j] + (vAe_i L_j - vAe_j L_i)}{\left(\frac{\varsigma_0}{1+R} qAe_j + vAe_j\right)\left(\frac{\varsigma_0 R}{1+R} qAe_j + L_j\right)}. \end{aligned}$$

By taking a sufficiently small $\varsigma_0 > 0$, we can set the first term of the numerator to be negligible for the second term, $vAe_i L_j - vAe_j L_i > 0$. Then, $\frac{p_1 Ae_i}{p_1 Ae_j} > \frac{p_1 (I-A) e_i}{p_1 (I-A) e_j}$ for $i \notin J$ and $j \in J$.

Inductively, suppose $\frac{p_t Ae_i}{p_t Ae_j} > \frac{p_t (I-A) e_i}{p_t (I-A) e_j}$ $(i \notin J, j \in J)$ for $t \geqq 1$. Since

$$\begin{aligned} 0 &< \frac{p_t Ae_i}{p_t Ae_j} - \frac{p_t (I-A)\, e_i}{p_t (I-A)\, e_j} = \frac{\frac{\varsigma_0}{(1+R)^t} qAe_i + vAe_i}{\frac{\varsigma_0}{(1+R)^t} qAe_j + vAe_j} - \frac{\frac{\varsigma_0 R}{(1+R)^t} qAe_i + L_i}{\frac{\varsigma_0 R}{(1+R)^t} qAe_j + L_j} \\ &= \frac{\frac{\varsigma_0}{(1+R)^t} qA\,[(L_j - RvAe_j)\, e_i - (L_i - RvAe_i)\, e_j] + (vAe_i L_j - vAe_j L_i)}{\left[\frac{\varsigma_0}{(1+R)^t} qAe_j + vAe_j\right]\left[\frac{\varsigma_0 R}{(1+R)^t} qAe_j + L_j\right]}, \end{aligned}$$

it follows that

$$\begin{aligned} vAe_iL_j - vAe_jL_i &> -\frac{\varsigma_0}{(1+R)^t}qA\left[(L_j - RvAe_j)\,e_i - (L_i - RvAe_i)\,e_j\right] \\ &> -\frac{\varsigma_0}{(1+R)^{t+1}}qA\left[(L_j - RvAe_j)\,e_i - (L_i - RvAe_i)\,e_j\right], \end{aligned}$$

even in the case where $qA\left[(L_j - RvAe_j)\,e_i - (L_i - RvAe_i)\,e_j\right] < 0$ holds. Therefore, in any case of $qA\left[(L_j - RvAe_j)\,e_i - (L_i - RvAe_i)\,e_j\right] \gtreqqless 0$, we have

$$\begin{aligned} &\frac{p_{t+1}Ae_i}{p_{t+1}Ae_j} - \frac{p_{t+1}\,(I-A)\,e_i}{p_{t+1}\,(I-A)\,e_j} \\ &= \frac{\frac{\varsigma_0}{(1+R)^{t+1}}qA\left[(L_j - RvAe_j)\,e_i - (L_i - RvAe_i)\,e_j\right] + (vAe_iL_j - vAe_jL_i)}{\left[\frac{\varsigma_0}{(1+R)^{t+1}}qAe_j + vAe_j\right]\left[\frac{\varsigma_0 R}{(1+R)^{t+1}}qAe_j + L_j\right]} > 0. \end{aligned}$$

By the construction, $vAe_iL_j = vAe_jL_i$ for $i, j \in J$. Note that there exists $j^* \in J$ such that for any $j \in J$, $qA\left[(L_{j^*} - RvAe_{j^*})\,e_j - (L_j - RvAe_j)\,e_{j^*}\right] \geqq 0$. Then, it holds that for any $t = 1, 2, \ldots$,

$$\frac{p_tAe_j}{p_tAe_{j^*}} - \frac{p_t\,(I-A)\,e_j}{p_t\,(I-A)\,e_{j^*}} = \frac{\frac{\varsigma_0}{(1+R)^t}qA\left[(L_{j^*} - RvAe_{j^*})\,e_j - (L_j - RvAe_j)\,e_{j^*}\right]}{\left[\frac{\varsigma_0}{(1+R)^t}qAe_{j^*} + vAe_{j^*}\right]\left[\frac{\varsigma_0 R}{(1+R)^t}qAe_{j^*} + L_{j^*}\right]} \geqq 0.$$

This completes the proof. □

Without loss of generality, let the commodity j in Lemma IV be renumbered as n, so that

$$\frac{vAe_i}{L_i} \geqq \frac{vAe_n}{L_n} \text{ for } i \neq n, \text{ and } \frac{vAe_i}{L_i} > \frac{vAe_n}{L_n} \text{ for some } i.$$

Then, Lemma IV implies that for every $t = 1, 2, \ldots$,

$$\frac{p_tAe_i}{p_tAe_n} \geqq \frac{p_t\,(I-A)\,e_i}{p_t\,(I-A)\,e_n} \text{ for } i \neq n, \text{ and } \frac{p_tAe_i}{p_tAe_n} > \frac{p_t\,(I-A)\,e_i}{p_t\,(I-A)\,e_n} \text{ for some } i.$$

As vA means the labor values of capital goods, this numbering makes the n-th commodity the most labor-intensive under the labor-value pricing.

The next lemma characterizes a necessary and sufficient condition for preserving the non-reversal of factor-intensity through the infinite limit of time. Such a condition is given by $\frac{qAe_i}{L_i} > \frac{qAe_n}{L_n}$ for $i \neq n$ as follows:

Lemma V: Take $p_0 = \tilde{p}_0(\varsigma_0)$ for an arbitrary $\varsigma_0 > 0$, and let $\{p_{t-1}; (w_t, r_t)\}_{t=1}^{\infty} = \mathbf{p}(p_0, w_0, \mathbf{r})$ for $\mathbf{r} \subset \overline{\mathbb{R}}_+$. Then, the following are equivalent: for $i = 1, \ldots, n-1$,

(i) $\frac{qAe_i}{L_i} > \frac{qAe_n}{L_n}$;

(ii) $\frac{p_t(I-A)e_i}{p_t(I-A)e_n} > \frac{p_{t+1}(I-A)e_i}{p_{t+1}(I-A)e_n}$ for all $t = 1, 2, \ldots$;

(iii) $\frac{p_t(I-A)e_i}{L_i} > \frac{p_t(I-A)e_n}{L_n}$ for all $t = 1, 2, \ldots$.

Proof. (i) ⇔ (ii): Fix t arbitrarily. We have from (A3) that, for any $i \neq n$,

$$\begin{aligned} & \frac{p_t(I-A)e_i}{p_t(I-A)e_n} > \frac{p_{t+1}(I-A)e_i}{p_{t+1}(I-A)e_n} \\ \Leftrightarrow \ & \frac{\frac{\varsigma_0 R}{(1+R)^t} qAe_i + w_0 L_i}{\frac{\varsigma_0 R}{(1+R)^t} qAe_n + w_0 L_n} > \frac{\frac{\varsigma_0 R}{(1+R)^{t+1}} qAe_i + w_0 L_i}{\frac{\varsigma_0 R}{(1+R)^{t+1}} qAe_n + w_0 L_n} \\ \Leftrightarrow \ & \frac{\varsigma_0 R^2 w_0}{(1+R)^{t+1}} (qAe_i L_n - qAe_n L_i) > 0. \end{aligned}$$

Then, $\frac{p_t(I-A)e_i}{p_t(I-A)e_n} > \frac{p_{t+1}(I-A)e_i}{p_{t+1}(I-A)e_n}$ for all $t = 1, 2, \ldots$, if and only if $qAe_i L_n > qAe_n L_i$.

(i) ⇔ (iii): Similarly, (A3) yields that for any $i \neq n$,

$$\begin{aligned} \frac{p_t(I-A)e_i}{p_t(I-A)e_n} > \frac{L_i}{L_n} \ & \Leftrightarrow \ \frac{\frac{\varsigma_0 R}{(1+R)^t} qAe_i + w_0 L_i}{\frac{\varsigma_0 R}{(1+R)^t} qAe_n + w_0 L_n} > \frac{L_i}{L_n} \\ & \Leftrightarrow \ \frac{\varsigma_0 R w_0}{(1+R)^t} (qAe_i L_n - qAe_n L_i) > 0. \end{aligned}$$

Then, $\frac{p_t(I-A)e_i}{p_t(I-A)e_n} > \frac{L_i}{L_n}$ for all $t = 1, 2, \ldots$, if and only if $qAe_i L_n > qAe_n L_i$. □

Given (A, L) with $\frac{qAe_i}{L_i} > \frac{qAe_n}{L_n}$ and $\frac{vAe_i}{L_i} \geqq \frac{vAe_n}{L_n}$ for $i = 1, \ldots, n-1$, and $\frac{vAe_i}{L_i} > \frac{vAe_n}{L_n}$ for some i. For $p_0 = \tilde{p}_0(\varsigma_0) > \mathbf{0}, w_0 > 0, \mathbf{r} \subset \overline{\mathbb{R}}_+$, let us take $\{p_{t-1}; (w_t, r_t)\}_{t=1}^{\infty} = \mathbf{p}(p_0, w_0, \mathbf{r})$ such that for each $t = 1, 2, \ldots$,

$$\begin{aligned} \frac{p_t A e_i}{p_t A e_n} &\geqq \frac{p_t (I-A) e_i}{p_t (I-A) e_n} > \frac{p_{t+1}(I-A) e_i}{p_{t+1}(I-A) e_n} \text{ for } i \neq n; \\ \frac{p_t A e_i}{p_t A e_n} &> \frac{p_t (I-A) e_i}{p_t (I-A) e_n} \text{ for some } i. \end{aligned}$$

Denote such $\{p_{t-1}; (w_t, r_t)\}_{t=1}^{\infty}$ by $\tilde{\mathbf{p}}(p_0, w_0, \mathbf{r})$. Note that $\tilde{\mathbf{p}}(p_0, w_0, \mathbf{r})$ is well-defined whenever $p_0 = \tilde{p}_0(\varsigma_0)$ by Lemmas IV and V.

Lemma III shows that $\left\{(x_t^{\nu}, Ax_t^{\nu})_{\nu \in \mathcal{N}}\right\}_{t=1}^{\infty}$ (from $\overline{\omega}$) may be an RS allocation. Such an allocation naturally meets constraints (1) and (3). We now check whether it satisfies the labor hours constraint (2), i.e., $Lx_t^{\nu} \in (0, 1]$ ($\forall \nu \in \mathcal{N}$; $\forall t = 1, 2, \ldots$). To this end, fix arbitrarily $t = 1, 2, \ldots$. Since $w_t = \prod_{\tau=1}^{t}(1 + r_\tau) w_0$ and $p_t = (1 + r_t) p_{t-1} A + w_t L$ with the arbitrage-free condition $(1 + r_t) p_{t-1} - p_t \geqq \mathbf{0}$, we have that, for any $x_t^{\nu} \in \mathbb{R}_+^n$ fulfilling $p_t (I - A) x_t^{\nu} = p_t b$,

$$\prod_{\tau=1}^{t}(1 + r_\tau) w_0 L x_t^{\nu} = p_t b - \left[(1 + r_t) p_{t-1} - p_t\right] A x_t^{\nu} \leqq p_t b.$$

which derives from $p_t = \prod_{\tau=1}^{t}(1 + r_\tau) p_0 A^t + \prod_{\tau=1}^{t}(1 + r_\tau) w_0 L \sum_{\tau=0}^{t} A^{\tau}$. Then,

$$Lx_t^{\nu} \leqq \frac{p_t b}{\prod_{\tau=1}^{t}(1 + r_\tau) w_0} = \frac{p_0}{w_0} A^t b + L \sum_{\tau=0}^{t} A^{\tau} b. \qquad \text{(A5)}$$

Thus, for $\omega_1^{\nu} \in \mathbb{R}_+^n$, a path $\{x_t^{\nu}\}_{t=1}^{\infty}$ with $Lx_t^{\nu} \in (0, 1]$ ($\forall t = 1, 2, \ldots$) is guaranteed by

$$\frac{p_0}{w_0} A^t b + L \sum_{\tau=0}^{t} A^{\tau} b \leqq 1, \qquad \text{(A6)}$$

for all $t = 1, 2, \ldots$. Since it may not be easy to evaluate the left-hand side of (A6) at any period t, we will suggest a simple sufficient condition for (A6).

Lemma VI: Suppose that (A, L, b) satisfies $b \geqq Ab$ and $(v + LA) b < 1$. For an arbitrary $w_0 > 0$, take $p_0 > \mathbf{0}$ such that $\frac{p_0}{w_0} = v + \varepsilon$, where ε is a

$1 \times n$ nonnegative vector such that $\varepsilon Ab \leqq 1-(v+L)\,b$. Then, it meets (A6) without equality for all $t=1,2,\ldots$, so that any $\{x_t^{\nu}\}_{t=1}^{\infty} \subset \mathbb{R}_+^n$ with $p_t\,(I-A)\,x_t^{\nu} = p_t b$ $(\forall t = 1,2,\ldots)$ fulfills $Lx_t^{\nu} \in (0,1)$ for all $t=1,2,\ldots$.

Proof. By the supposition of $1-(v+LA)\,b>0$, we can take a sufficiently small number $\varsigma_0>0$ with $\varepsilon \equiv \varsigma_0 q$ such that $\varepsilon Ab \leqq 1-(v+LA)\,b$ holds. Let $\varepsilon \equiv \frac{\varsigma_0 q}{w_0}$ and $\frac{p_0}{w_0} \equiv v+\varepsilon$ for an arbitrary $w_0>0$. Then,

$$\begin{aligned} p_0 - p_0 A &= w_0\,(v+\varepsilon)\,(I-A) = w_0 v\,(I-A) + w_0\varepsilon\,(I-A) = w_0 L + w_0\varsigma_0 q\,(I-A) \\ &> w_0 L \end{aligned}$$

as $q\,(I-A) = \frac{R}{1+R} q > \mathbf{0}$. In this case, by the construction of ε and p_0,

$$\frac{p_0}{w_0} Ab + L\,(I+A)\,b = vAb + \varepsilon Ab + Lb + LAb = vb + LAb + \varepsilon Ab \leqq 1.$$

Therefore, (A6) is satisfied for $t=1$. Moreover, by the supposition of $b \geqq Ab$, it follows that for any $t=1,2,\ldots$,

$$\begin{aligned} & \left[\frac{p_0}{w_0} A^{t-1} + L\sum_{\tau=0}^{t-1} A^{\tau}\right] b - \left[\frac{p_0}{w_0} A^{t} + L\sum_{\tau=0}^{t} A^{\tau}\right] b \\ = {} & \frac{p_0}{w_0} A^{t-1}\,(I-A)\,b + L\sum_{\tau=0}^{t-1} A^{\tau} b - \left(L\sum_{\tau=0}^{t-1} A^{\tau} b + LA^t b\right) \\ = {} & \left[\frac{p_0}{w_0} A^{t-1}\,(I-A) - LA^t\right] b = \left[\frac{p_0}{w_0}\,(I-A)\,A^{t-1} - LA^t\right] b = \left[\frac{p_0}{w_0}\,(I-A) - LA\right] A^{t-1} b \\ > {} & \left[v\,(I-A) - LA\right] A^{t-1} b \quad \left(\text{by } \frac{p_0}{w_0} \equiv v+\varepsilon \text{ with } \varepsilon > \mathbf{0}\right) \\ = {} & L\,(I-A)\,A^{t-1} b = LA^{t-1}\,(I-A)\,b \geqq 0. \end{aligned}$$

This implies that for any $t=2,\ldots$,

$$\frac{p_0}{w_0} A^t b + L\sum_{\tau=0}^{t} A^{\tau} b \leqq \frac{p_0}{w_0} Ab + L\,(I+A)\,b \leqq 1,$$

which implies that (A6) is satisfied for $t=2,\ldots$, and thus completes the proof. □

Note that, for the pricing paths starting with $p_0 = \tilde{p}_0(\varsigma_0)$ defined at the beginning of this subsection, we can apply Lemma VI by taking a sufficiently small $\varsigma_0 > 0$. Indeed, it suffices to set $\varsigma_0 q \leqq w_0\varepsilon$. Thus, the tractable price paths $\tilde{\mathbf{p}}(p_0, w_0, \mathbf{r})$ satisfies $\frac{p_0}{w_0}Ab + vb \leqq 1$ and so every ν's labor hours constraint at every period.

We now prove the existence of RSs. Given $\overline{\omega} = \omega^{Nb}$, let

$$X \equiv \left\{ (x^\nu)_{\nu\in\mathcal{N}} \in \mathbb{R}^{nN}_{++} \mid Ax^\nu = \omega^{Nb} \;\; (\forall\nu\in\mathcal{N}) \text{ and } Lx^\nu \in (0,1) \right\}.$$

The following proposition establishes the existence of each nation's catching-up optimal paths for the suitable initial endowments profile (supposed to exist, tentatively.)

Proposition A.4: Let $\langle \mathcal{N}, (A, L), u, \overline{\omega} \rangle$ with $\overline{\omega} = \omega^{Nb}$ be an economy satisfying $\frac{qAe_i}{L_i} > \frac{qAe_n}{L_n}$, $\frac{vAe_i}{L_i} \geqq \frac{vAe_n}{L_n}$ for $i = 1, \ldots, n-1$, $\frac{vAe_i}{L_i} > \frac{vAe_n}{L_n}$ for some i, $b \geqq Ab$, and $(v + LA)b < 1$. For $w_0 = 1$ and an arbitrary $\mathbf{r} \subset \overline{\mathbb{R}}_+$, take p_0 such that $p_0Ab + L(I + A)b \leqq 1$ and $\tilde{\mathbf{p}}(p_0, 1, \mathbf{r})$ is well-defined, and let $\{p_{t-1}; (w_t, r_t)\}_{t=1}^\infty = \tilde{\mathbf{p}}(p_0, 1, \mathbf{r})$. If there exist $(\omega_1^\nu)_{\nu\in\mathcal{N}} \in \mathbb{R}^{nN}_+$ and $(x_1^\nu)_{\nu\in\mathcal{N}} \in X$ such that $p_0Ax_1^\nu = p_0\omega_1^\nu$ and $p_1(I - A)x_1^\nu = p_1b$ for every $\nu \in \mathcal{N}$, then it constitutes $\left\{(x_t^\nu)_{\nu\in\mathcal{N}}\right\}_{t=1}^\infty \subset X$ such that $\{x_t^\nu\}_{t=1}^\infty \in \mathbf{x}(\omega_1^\nu)$ is catching-up optimal for every $\nu \in \mathcal{N}$. It thereby guarantees the existence of RS allocations $\left\{(x_t^{*\nu}, Ax_t^{*\nu})_{\nu\in\mathcal{N}}\right\}_{t=1}^\infty$ associated with $\{p_{t-1}; (w_t, r_t)\}_{t=1}^\infty$.

Proof. The suppositions, $\frac{qAe_i}{L_i} > \frac{qAe_n}{L_n}$ and $\frac{vAe_i}{L_i} \geqq \frac{vAe_n}{L_n}$ for $i = 1, \ldots, n-1$, $\frac{vAe_i}{L_i} > \frac{vAe_n}{L_n}$ for some i, and $b \geqq Ab$, and $(v + LA)b < 1$, ensure the existence of p_0 such that $p_0Ab + L(I + A)b \leqq 1$ and $\tilde{\mathbf{p}}(p_0, 1, \mathbf{r})$ is well-defined for an arbitrary $\mathbf{r} \subset \overline{\mathbb{R}}$, by Lemmas IV–VI. Given $\{p_{t-1}; (w_t, r_t)\}_{t=1}^\infty = \tilde{\mathbf{p}}(p_0, 1, \mathbf{r})$, we will show that for $(x_t^\nu)_{\nu\in\mathcal{N}} \in X$ with $p_t(I - A)x_t^\nu = p_tb$ $(\forall\nu \in \mathcal{N})$, there exists $\left(x_{t+1}^\nu\right)_{\nu\in\mathcal{N}} \in X$ such that $p_tAx_{t+1}^\nu = p_tAx_t^\nu$ and $p_{t+1}(I - A)x_{t+1}^\nu \geqq p_{t+1}b$ $(\forall\nu \in \mathcal{N})$. First, by Lemma IV, $\frac{p_tAe_i}{p_tAe_n} \geqq \frac{p_t(I-A)e_i}{p_t(I-A)e_n}$ holds for $i \neq n$, and $\frac{p_tAe_i}{p_tAe_n} > \frac{p_t(I-A)e_i}{p_t(I-A)e_n}$ holds for some i. This property and $(x_t^\nu)_{\nu\in\mathcal{N}} \in X \subset \mathbb{R}^{nN}_{++}$ together imply that, for any $\nu \in \mathcal{N}$,

$$\frac{p_tAx_t^\nu}{p_tAe_n} = \sum_{i=1}^{n-1}\left(\frac{p_tAe_i}{p_tAe_n}x_{it}^\nu\right) + x_{nt}^\nu > \sum_{i=1}^{n-1}\left[\frac{p_t(I-A)e_i}{p_t(I-A)e_n}x_{it}^\nu\right] + x_{nt}^\nu = \frac{p_t(I-A)x_t^\nu}{p_t(I-A)e_n}.$$

Then, since $p_t(I - A)x_t^\nu = p_tb$, $\frac{p_tAe_i}{p_tAe_n} \geqq \frac{p_t(I-A)e_i}{p_t(I-A)e_n}$ for $i \neq n$, and $\frac{p_tAe_i}{p_tAe_n} >$

$\frac{p_t(I-A)e_i}{p_t(I-A)e_n}$ for some i, we have:

$$\begin{aligned}\frac{p_t\left(I-A\right)x_t^{\nu}}{p_t\left(I-A\right)e_n}=\frac{p_t\left(I-A\right)x}{p_t\left(I-A\right)e_n}&=\sum_{i=1}^{n-1}\left[\frac{p_t\left(I-A\right)e_i}{p_t\left(I-A\right)e_n}x_i\right]+x_n\\&>\sum_{i=1}^{n-1}\left[\frac{p_{t+1}\left(I-A\right)e_i}{p_{t+1}\left(I-A\right)e_n}x_i\right]+x_n \text{ (by Lemma V)}\\&=\frac{p_{t+1}b}{p_{t+1}\left(I-A\right)e_n}\end{aligned}$$

for $x\equiv\left(I-A\right)^{-1}b$ with the i-th element x_i, so that $\frac{p_tAx_t^{\nu}}{p_tAe_n}>\frac{p_{t+1}b}{p_{t+1}(I-A)e_n}$ for every $\nu\in\mathcal{N}$. Defining $\hat{x}_{t+1}\equiv\frac{p_tAx_t^{\nu}}{p_tAe_n}e_n$, we get $p_tA\hat{x}_{t+1}=p_tAx_t^{\nu}$ and $p_{t+1}\left(I-A\right)\hat{x}_{t+1}>p_{t+1}b$. For $\varepsilon>0$, there exists $x_{t+1}\in\left\{y\in\mathbb{R}_{++}^n \mid \|y-\hat{x}_{t+1}\|<\varepsilon\right\}$ such that $p_tAx_{t+1}=p_tAx_t^{\nu}$ and $p_{t+1}\left(I-A\right)x_{t+1}\geqq p_{t+1}b$, where $\|\cdot\|$ is the standard Euclidean norm. Then, the program (MP_{t+1}^{ν}) has a solution since its domain is non-empty for $x_t^{\nu}\in\mathbb{R}_{++}^n$ with $p_t\left(I-A\right)x_t^{\nu}=p_tb$.

Suppose that $(\omega_1^{\nu})_{\nu\in\mathcal{N}}\in\mathbb{R}_+^{nN}$ and $(x_1^{\nu})_{\nu\in\mathcal{N}}\in X$ fulfill $p_0Ax_1^{\nu}=p_0\omega_1^{\nu}$ and $p_1\left(I-A\right)x_1^{\nu}=p_1b$ $(\forall\nu\in\mathcal{N})$. Then, for an arbitrary $\nu\in\mathcal{N}$, there exists a solution to the program (MP_1^{ν}), $x_1^{*\nu}\in\mathbb{R}_{++}^n$ with $\omega_2^{\nu}\equiv Ax_1^{*\nu}$, which meets $p_1\left(I-A\right)x_1^{*\nu}=p_1b$. Inductively, for $t\geqq 2$, we can apply Lemma 2 in order to conclude that $x_{t-1}^{*\nu}\in\mathbb{R}_{++}^n$ with $p_{t-1}\left(I-A\right)x_{t-1}^{*\nu}=p_{t-1}b$ guarantees the existence of a solution to the program (MP_t^{ν}). Let us denote this solution as $x_t^{*\nu}\in\mathbb{R}_{++}^n$ with $\omega_{t+1}^{\nu}\equiv Ax_t^{*\nu}$. Then, such $x_t^{*\nu}$ further yields the existence of $x_{t+1}^{*\nu}\in\mathbb{R}_{++}^n$ with $p_{t+1}\left(I-A\right)x_{t+1}^{*\nu}=p_{t+1}b$. As $p_0Ab+vb\leqq 1$ ensures that $p_t\left(I-A\right)x_t^{*\nu}=p_tb$ implies $Lx_t^{*\nu}\in(0,1)$ $(\forall t=1,2,\ldots)$ by Lemma VI, we have $\{x_t^{*\nu}\}_{t=1}^{\infty}\in\mathbf{x}\left(\omega_1^{\nu}\right)$. By Corollary 2, $\{x_t^{*\nu}\}_{t=1}^{\infty}\in\mathbf{x}\left(\omega_1^{\nu}\right)$, which is constituted inductively as argued above, is catching-up optimal for $\nu\in\mathcal{N}$, i.e., it satisfies the RS condition (a) of Definition 1. We note that this approach to constitute an catching-up optimal path $\{x_t^{*\nu}\}_{t=1}^{\infty}$ is still admissible when $\left\{(x_t^{*\nu})_{\nu\in\mathcal{N}}\right\}_{t=1}^{\infty}\subset X$ is additionally assumed. Then, we easily verify the RS conditions (b) and (c) since $Ax_t^{*\nu}=\omega_{t+1}^{\nu}$ $(\forall\nu\in\mathcal{N},\forall t=1,2,\ldots)$ and $(x_t^{*\nu})_{\nu\in\mathcal{N}},\left(x_{t+1}^{*\nu}\right)_{\nu\in\mathcal{N}}\in X$. This completes the proof. □

Here, we consider conditions on the wealth profile that characterize the initial endowments profile assumed in Proposition A.4. Given $\overline{\omega}=\omega^{Nb}$, we can take $\overline{p}_0$ such that $\overline{p}_0\geqq\overline{p}_0A+L>\mathbf{0}$ and $\frac{1}{\overline{p}_0\omega^{Nb}}\overline{p}_0Ab+vb\leqq 1$. Indeed, we have already argued that there exists $\overline{p}_0=\tilde{p}_0\left(\varsigma_0\right)$, which is defined by taking a sufficiently small $\varsigma_0 > 0$ in order to meet $\varsigma_0 q \leqq \varepsilon$, such that $\overline{p}_0 \geqq \overline{p}_0 A + L$ and $\overline{p}_0 Ab + vb \leqq 1$ hold by Lemma VI. Moreover, by the definition of $\omega^{Nb} = A(I-A)^{-1} Nb$, $\frac{1}{\overline{p}_0 \omega^{Nb}} \overline{p}_0 Ab < \overline{p}_0 Ab$ holds.

Let $p_1^r \equiv (1+r)\overline{p}_0 A + (1+r) L$ for $r \geqq 0$. For each $\theta \in [0,1]$, we define the set

$$X^r(\theta) \equiv \left\{ x \in \mathbb{R}_+^n \mid \overline{p}_0 Ax = \theta \overline{p}_0 \omega^{Nb},\ p_1^r (I-A)x = p_1^r b,\ Lx \in (0,1) \right\}$$

with generic element $x^r(\theta)$. Note that $X^r(\theta)$ is non-empty for some $\theta \in [0,1]$, since $(I-A)^{-1} b \in X^r\left(\frac{1}{N}\right)$. Indeed, $Ax^r\left(\frac{1}{N}\right) = A(I-A)^{-1} b = \frac{1}{N}\omega^{Nb}$ holds, so that

$$\overline{p}_0 Ax^r(\theta) = \overline{p}_0 Ax^r\left(\frac{1}{N}\right) = \frac{1}{N}\overline{p}_0 \omega^{Nb} = \theta \overline{p}_0 \omega^{Nb}.$$

Also, $(I-A)x^r\left(\frac{1}{N}\right) = b$ implies $p_1^r (I-A)x^r\left(\frac{1}{N}\right) = p_1^r b$. Finally, $L(I-A)^{-1}(Nb) \in (0,N)$ implies $\frac{1}{N}L(I-A)^{-1}(Nb) \in (0,1)$, so that

$$Lx^r(\theta) = Lx^r\left(\frac{1}{N}\right) = L(I-A)^{-1} b \in (0,1).$$

Thus, $X^r(\theta) \neq \varnothing$ for some $\theta \in [0,1)$.

Note that $X^r(\theta)$ is a compact set of $\mathbb{R}_+^n$ for any $\theta \in [0,1]$ since $\overline{p}_0 > \mathbf{0}$ and $p_1^r > \mathbf{0}$. Then, we can take $l^r(\theta) \equiv \min\left\{ Lx^r(\theta) \mid x^r(\theta) \in X^r(\theta) \right\}$ for each $\theta \in [0,1]$ satisfying $X^r(\theta) \neq \varnothing$, so that $l^r(\theta) \in (0,1]$. We define

$$\theta^r \equiv \inf\left\{ \theta \in [0,1] \mid l^r(\theta) \in (0,1] \right\},$$

where $\theta^r > 0$ by $b > \mathbf{0}$. We also define $\underline{\theta} \equiv \inf_{r \geqq 0} \theta^r > 0$. Then, consider the set:

$$\underline{\Delta}(W_0) \equiv \left\{ (W_0^\nu)_{\nu \in \mathcal{N}} \in \mathbb{R}_{++}^N \mid \sum_{\nu \in \mathcal{N}} W_0^\nu = 1,\ \min_{\nu \in \mathcal{N}} W_0^\nu > \underline{\theta} \right\}.$$

Moreover, let $\Delta(\overline{\omega}) \equiv \{ p > \mathbf{0} \mid p\overline{\omega} = 1 \}$.

Theorem A.1: Let $\langle \mathcal{N}, (A,L), u, \overline{\omega} \rangle$ with $\overline{\omega} = \omega^{Nb}$ be an economy satisfying $\frac{qAe_i}{L_i} > \frac{qAe_n}{L_n}$ and $\frac{vAe_i}{L_i} \geqq \frac{vAe_n}{L_n}$ for $i = 1, \ldots, n-1$, $\frac{vAe_i}{L_i} > \frac{vAe_n}{L_n}$ for some i, $b \geqq Ab$, and $(v + LA)b < 1$. Let $\left\{ \left(x_t^*, \delta_t^*, \omega_{t+1}^* \right) \right\}_{t=1}^{\infty}$ satisfy $Lx_t^* = Nvb$ for all $t = 1, 2, \ldots$. Moreover, let $(W_0^\nu)_{\nu \in \mathcal{N}} \in \underline{\Delta}(W_0)$ be a profile of initial wealth endowments. Then, there exist $(p_0^*, w_0^*) \in \Delta(\overline{\omega}) \times \mathbb{R}_{++}$, $\mathbf{r}^* \subset \overline{\mathbb{R}}_+$, and $(\omega_1^{*\nu})_{\nu \in \mathcal{N}} \in \mathbb{R}_+^{nN}$ satisfying $p_0^* \omega_1^{*\nu} = W_0^\nu$ by which $\left\{ \left(x_t^{*\nu}, \delta_t^{*\nu}, \omega_{t+1}^{*\nu} \right)_{\nu \in \mathcal{N}} \right\}_{t=1}^{\infty}$ constitutes an RS with $\left\{ p_{t-1}^*; (w_t^*, r_t^*) \right\}_{t=1}^{\infty} = \tilde{\mathbf{p}}(p_0^*, w_0^*, \mathbf{r}^*)$.

Proof. Let $\left\{ \left(x_t^*, \delta_t^*, \omega_{t+1}^* \right) \right\}_{t=1}^{\infty}$ satisfy $Lx_t^* = Nvb$ for all $t = 1, 2, \ldots$. Proposition A.1 implies that, for each $t = 1, 2, \ldots$, any assignment $(x_t^{*\nu})_{\nu \in \mathcal{N}} \in \mathbb{R}_+^{nN}$ satisfying $\sum_{\nu \in \mathcal{N}} x_t^{*\nu} = x_t^*$ and $Lx_t^{*\nu} \in [0, 1]$ $(\forall \nu \in \mathcal{N})$ can constitute a Pareto efficient allocation. Note that $x_t^* = (I - A)^{-1} (Nb)$ and $\delta_t^* = \mathbf{0}$ for all $t = 1, 2, \ldots$. Then, x_1^* satisfies $pAx_1^* = p\omega^{Nb}$ and $px_1^* = Npb + p\omega^{Nb}$ for any $p \geqq \mathbf{0}$ by an argument similar to the above case for $x^r\left(\frac{1}{N}\right) \in X^r\left(\frac{1}{N}\right)$.

Let $(W_0^\nu)_{\nu \in \mathcal{N}} \in \underline{\Delta}(W_0)$. By the construction, there exists $r \geqq 0$ such that $\min_{\nu \in \mathcal{N}} W_0^\nu \geqq \theta^r$, and there exists $x(r) \in X^r(\theta^r)$ with $\overline{p}_0$ satisfying $\overline{p}_0 \geqq \overline{p}_0 A + L > \mathbf{0}$ and $\frac{1}{\overline{p}_0 \omega^{Nb}} \overline{p}_0 Ab + vb \leqq 1$, and $p_1^r \equiv (1 + r)\overline{p}_0 A + (1 + r) L$. Define $p_0^* \equiv \frac{1}{\overline{p}_0 \omega^{Nb}} \overline{p}_0 \in \Delta\left(\omega^{Nb}\right)$. Then, there exists $(\theta^\nu)_{\nu \in \mathcal{N}} \in \mathbb{R}_{++}^N$ such that $\left(\theta^\nu \omega^{Nb}\right)_{\nu \in \mathcal{N}} \in \mathbb{R}_+^{nN}$ and $W_0^\nu = \theta^\nu p_0^* \omega^{Nb}$ $(\forall \nu \in \mathcal{N})$, and thus, we can take $(\omega_1^{*\nu})_{\nu \in \mathcal{N}} \in \mathbb{R}_+^{nN}$ such that $p_0^* \omega_1^{*\nu} = W_0^\nu$ $(\forall \nu \in \mathcal{N})$. For each $\nu \in \mathcal{N}$, let $x_1^{*\nu} \in \mathbb{R}_+^n$ be a solution to:

$$\min_{x \in \mathbb{R}_+^n} Lx, \text{ s.t. } p_0^* Ax = \theta^\nu p_0^* \omega^{Nb} = p_0^* \omega_1^{*\nu};\ p_1^*(I - A)x = p_1^* b,$$

where $p_1^* \equiv \frac{1}{\overline{p}_0 \omega^{Nb}} p_1^r = (1 + r) p_0^* A + \frac{1}{\overline{p}_0 \omega^{Nb}} (1 + r) L$. As we have seen in the proof of Proposition A.4, for each $t = 1, 2, \ldots$, there exist $(x_t^{*\nu})_{\nu \in \mathcal{N}}$ and $\left(x_{t+1}^{*\nu}\right)_{\nu \in \mathcal{N}} \in X$ such that $p_{t-1}^r Ax_{t+1}^{*\nu} = p_{t-1}^r Ax_t^{*\nu}$ and $p_t^r (I - A) x_t^{*\nu} = p_t^r b$ for $\left\{ p_{t-1}^r; (w_t, r_t^*) \right\}_{t=1}^{\infty} = \mathbf{p}(p_0^r, 1, \mathbf{r}^*)$, where $p_0^r \equiv \overline{p}_0$ and $\mathbf{r}^* \subset \overline{\mathbb{R}}_+$ starts with $r_1^* = r$. Then, taking $\left\{ p_{t-1}^*; (w_t^*, r_t^*) \right\}_{t=1}^{\infty} = \mathbf{p}(p_0^*, w_0^*, \mathbf{r}^*)$ for $w_0^* \equiv \frac{1}{\overline{p}_0 \omega^{Nb}} > 0$, we obtain $\left\{ (x_t^{*\nu})_{\nu \in \mathcal{N}} \right\}_{t=1}^{\infty} \subset X$ such that $p_0^* Ax_1^{*\nu} = p_0^* \omega_1^{*\nu}$, $p_t^* Ax_{t+1}^{*\nu} = p_t^* Ax_t^{*\nu}$, and $p_t^* (I - A) x_t^{*\nu} = p_t^* b$ for all $t = 1, 2, \ldots$. Let $\left\{ \left(\delta_t^{*\nu}, \omega_{t+1}^{*\nu} \right)_{\nu \in \mathcal{N}} \right\}_{t=1}^{\infty}$ consist of $\delta_t^{*\nu} = \mathbf{0}$ and $\omega_{t+1}^{*\nu} \equiv Ax_t^{*\nu}$ $(\forall \nu \in \mathcal{N}, \forall t = 1, 2, \ldots)$. Then, $\left\{ \left(x_t^{*\nu}, \delta_t^{*\nu}, \omega_{t+1}^{*\nu} \right)_{\nu \in \mathcal{N}} \right\}_{t=1}^{\infty}$ constitutes an RS with $\left\{ p_{t-1}^*; (w_t^*, r_t^*) \right\}_{t=1}^{\infty}$ by Proposition A.4 again. □

Remark A.1: Note that in Theorem A1, there is no constraint on $\mathbf{r}^* \subset \overline{\mathbb{R}}_+$ except for the initial interest rate r_1^*. Therefore, this theorem allows the existence of an RS in which $\liminf_{t \to \infty} r_t^* > R$ holds.

8 Alternative statements of Lemma VI and Proposition A4

Lemma VI: Suppose that (A, L, b) satisfies $v(I + A)b < 1$. For an arbitrary $w_0 > 0$, take $p_0 > \mathbf{0}$ such that $\frac{p_0}{w_0} = v + \varepsilon$, where ε is a $1 \times n$ nonnegative vector such that $\varepsilon Ab \leqq 1 - v(I + A)b$. Then, it meets (A6) without equality for all $t = 1, 2, \ldots$, so that any $\{x_t^\nu\}_{t=1}^\infty \subset \mathbb{R}_+^n$ with $p_t(I - A)x_t^\nu = p_t b$ $(\forall t = 1, 2, \ldots)$ fulfills $Lx_t^\nu \in (0, 1)$ for all $t = 1, 2, \ldots$.

Proof. By the supposition of $1 - v(I + A)b > 0$, we can take a $1 \times n$ vector $\varepsilon \geqq \mathbf{0}$ (possibly nonzero) such that $\varepsilon Ab \leqq 1 - v(I + A)b$. Putting $p_0 \equiv w_0(v + \varepsilon)$ for an arbitrary $w_0 > 0$, we have $p_0 \geqq p_0 A + w_0 L > p_0 A$ following from $L > \mathbf{0}$. By the construction of ε and p_0,

$$\frac{p_0}{w_0}Ab + vb = vAb + \varepsilon Ab + vb = v(I + A)b + \varepsilon Ab \leqq 1.$$

Moreover, $p_0 A^t b$ is decreasing with respect to t since for any t,

$$p_0 A^t b - p_0 A^{t+1} b = p_0 (I - A) A^t b > 0$$

following from $p_0(I - A) > \mathbf{0}$ and $A^t b > \mathbf{0}$. Additionally, noting that $(I - A)^{-1} = \sum_{\tau=0}^\infty A^\tau \geq \sum_{\tau=0}^t A^\tau$ for any t, we obtain that for any $t = 1, 2, \ldots$,

$$\frac{p_0}{w_0}A^t b + vb < \frac{p_0}{w_0}Ab + vb \leqq 1,$$

which completes the proof. □

Proposition A.4: Let $\langle \mathcal{N}, (A, L), u, \overline{\omega} \rangle$ with $\overline{\omega} = \omega^{Nb}$ be an economy satisfying $\frac{qAe_i}{L_i} > \frac{qAe_n}{L_n}$ and $\frac{vAe_i}{L_i} \geqq \frac{vAe_n}{L_n}$ for $i = 1, \ldots, n-1$, $\frac{vAe_i}{L_i} > \frac{vAe_n}{L_n}$ for some i, and $v(I + A)b < 1$. For $w_0 = 1$ and an arbitrary $\mathbf{r} \subset \overline{\mathbb{R}}_+$, take p_0 such that $p_0 Ab + vb \leqq 1$ and $\tilde{\mathbf{p}}(p_0, 1, \mathbf{r})$ is well-defined, and let $\{p_{t-1}; (w_t, r_t)\}_{t=1}^\infty = \tilde{\mathbf{p}}(p_0, 1, \mathbf{r})$. If there exist $(\omega_1^\nu)_{\nu \in \mathcal{N}} \in \mathbb{R}_+^{nN}$ and $(x_1^\nu)_{\nu \in \mathcal{N}} \in X$ such that $p_0 A x_1^\nu = p_0 \omega_1^\nu$ and $p_1(I - A)x_1^\nu = p_1 b$ for every $\nu \in \mathcal{N}$, then it constitutes $\left\{(x_t^\nu)_{\nu \in \mathcal{N}}\right\}_{t=1}^\infty \subset X$ such that $\{x_t^\nu\}_{t=1}^\infty \in \mathbf{x}(\omega_1^\nu)$ is catching-up optimal for every $\nu \in \mathcal{N}$. It thereby guarantees the existence of RS allocations $\left\{(x_t^{*\nu}, Ax_t^{*\nu})_{\nu \in \mathcal{N}}\right\}_{t=1}^\infty$ associated with $\{p_{t-1}; (w_t, r_t)\}_{t=1}^\infty$.

Proof. The suppositions, $\frac{qAe_i}{L_i} > \frac{qAe_n}{L_n}$ and $\frac{vAe_i}{L_i} \geqq \frac{vAe_n}{L_n}$ for $i = 1, \ldots, n-1$, $\frac{vAe_i}{L_i} > \frac{vAe_n}{L_n}$ for some i, and $v(I+A)b < 1$, ensure the existence of p_0 such that $p_0 Ab + vb \leqq 1$ and $\tilde{\mathbf{p}}(p_0, 1, \mathbf{r})$ is well-defined for an arbitrary $\mathbf{r} \subset \overline{\mathbb{R}}$, by Lemmas IV–VI. Given $\{p_{t-1}; (w_t, r_t)\}_{t=1}^{\infty} = \tilde{\mathbf{p}}(p_0, 1, \mathbf{r})$, we will show that for $(x_t^\nu)_{\nu \in \mathcal{N}} \in X$ with $p_t(I-A)x_t^\nu = p_t b$ $(\forall \nu \in \mathcal{N})$, there exists $\left(x_{t+1}^\nu\right)_{\nu \in \mathcal{N}} \in X$ such that $p_t A x_{t+1}^\nu = p_t A x_t^\nu$ and $p_{t+1}(I-A)x_{t+1}^\nu \geqq p_{t+1} b$ $(\forall \nu \in \mathcal{N})$. First, by Lemma IV, $\frac{p_t Ae_i}{p_t Ae_n} \geqq \frac{p_t(I-A)e_i}{p_t(I-A)e_n}$ holds for $i \neq n$, and $\frac{p_t Ae_i}{p_t Ae_n} > \frac{p_t(I-A)e_i}{p_t(I-A)e_n}$ holds for some i. This property and $(x_t^\nu)_{\nu \in \mathcal{N}} \in X \subset \mathbb{R}_{++}^{nN}$ together imply that, for any $\nu \in \mathcal{N}$,

$$\frac{p_t A x_t^\nu}{p_t A e_n} = \sum_{i=1}^{n-1} \left(\frac{p_t A e_i}{p_t A e_n} x_{it}^\nu \right) + x_{nt}^\nu > \sum_{i=1}^{n-1} \left[\frac{p_t (I-A) e_i}{p_t (I-A) e_n} x_{it}^\nu \right] + x_{nt}^\nu = \frac{p_t (I-A) x_t^\nu}{p_t (I-A) e_n}.$$

Then, since $p_t(I-A)x_t^\nu = p_t b$, $\frac{p_t Ae_i}{p_t Ae_n} \geqq \frac{p_t(I-A)e_i}{p_t(I-A)e_n}$ for $i \neq n$, and $\frac{p_t Ae_i}{p_t Ae_n} > \frac{p_t(I-A)e_i}{p_t(I-A)e_n}$ for some i, we have:

$$\begin{aligned}
\frac{p_t (I-A) x_t^\nu}{p_t (I-A) e_n} = \frac{p_t (I-A) x}{p_t (I-A) e_n} &= \sum_{i=1}^{n-1} \left[\frac{p_t (I-A) e_i}{p_t (I-A) e_n} x_i \right] + x_n \\
&> \sum_{i=1}^{n-1} \left[\frac{p_{t+1} (I-A) e_i}{p_{t+1} (I-A) e_n} x_i \right] + x_n \text{ (by Lemma V)} \\
&= \frac{p_{t+1} b}{p_{t+1} (I-A) e_n}
\end{aligned}$$

for $x \equiv (I-A)^{-1} b$ with the i-th element x_i, so that $\frac{p_t A x_t^\nu}{p_t A e_n} > \frac{p_{t+1} b}{p_{t+1}(I-A)e_n}$ for every $\nu \in \mathcal{N}$. Defining $\hat{x}_{t+1} \equiv \frac{p_t A x_t^\nu}{p_t A e_n} e_n$, we get $p_t A \hat{x}_{t+1} = p_t A x_t^\nu$ and $p_{t+1}(I-A)\hat{x}_{t+1} > p_{t+1} b$. For $\varepsilon > 0$, there exists $x_{t+1} \in \left\{ y \in \mathbb{R}_{++}^n \mid \|y - \hat{x}_{t+1}\| < \varepsilon \right\}$ such that $p_t A x_{t+1} = p_t A x_t^\nu$ and $p_{t+1}(I-A)x_{t+1} \geqq p_{t+1} b$, where $\|\cdot\|$ is the standard Euclidean norm. Then, the program (MP_{t+1}^ν) has a solution since its domain is non-empty for $x_t^\nu \in \mathbb{R}_{++}^n$ with $p_t(I-A)x_t^\nu = p_t b$.

Suppose that $(\omega_1^\nu)_{\nu \in \mathcal{N}} \in \mathbb{R}_+^{nN}$ and $(x_1^\nu)_{\nu \in \mathcal{N}} \in X$ fulfill $p_0 A x_1^\nu = p_0 \omega_1^\nu$ and $p_1(I-A)x_1^\nu = p_1 b$ $(\forall \nu \in \mathcal{N})$. Then, for an arbitrary $\nu \in \mathcal{N}$, there exists a solution to the program (MP_1^ν), $x_1^{*\nu} \in \mathbb{R}_{++}^n$ with $\omega_2^\nu \equiv A x_1^{*\nu}$, which meets $p_1(I-A)x_1^{*\nu} = p_1 b$. Inductively, for $t \geqq 2$, we can apply Lemma 2 in order to conclude that $x_{t-1}^{*\nu} \in \mathbb{R}_{++}^n$ with $p_{t-1}(I-A)x_{t-1}^{*\nu} = p_{t-1} b$ guarantees the existence of a solution to the program (MP_t^ν). Let us denote this solution as $x_t^{*\nu} \in \mathbb{R}_{++}^n$ with $\omega_{t+1}^{\nu} \equiv Ax_t^{*\nu}$. Then, such $x_t^{*\nu}$ further yields the existence of $x_{t+1}^{*\nu} \in \mathbb{R}_{++}^n$ with $p_{t+1}(I-A)x_{t+1}^{*\nu} = p_{t+1}b$. As $p_0Ab + vb \leqq 1$ ensures that $p_t(I-A)x_t^{*\nu} = p_tb$ implies $Lx_t^{*\nu} \in (0,1)$ $(\forall t = 1, 2, \ldots)$ by Lemma VI, we have $\{x_t^{*\nu}\}_{t=1}^{\infty} \in \mathbf{x}(\omega_1^{\nu})$. By Corollary 2, $\{x_t^{*\nu}\}_{t=1}^{\infty} \in \mathbf{x}(\omega_1^{\nu})$, constituted inductively as argued above, is catching-up optimal for $\nu \in \mathcal{N}$, i.e., it satisfies the RS condition (a) of Definition 1. We note that this approach to constitute a catching-up optimal path $\{x_t^{*\nu}\}_{t=1}^{\infty}$ is still admissible when $\left\{(x_t^{*\nu})_{\nu\in\mathcal{N}}\right\}_{t=1}^{\infty} \subset X$ is additionally assumed. Then, we easily verify that the RS conditions (b) and (c) since $Ax_t^{*\nu} = \omega_{t+1}^{\nu}$ $(\forall \nu \in \mathcal{N}, \forall t = 1, 2, \ldots)$ and $(x_t^{*\nu})_{\nu\in\mathcal{N}}, \left(x_{t+1}^{*\nu}\right)_{\nu\in\mathcal{N}} \in X$. This completes the proof. □

We here consider conditions on the wealth profile that characterize the initial endowments profile supposed in Proposition A.4. Given $\overline{\omega} = \omega^{Nb}$, we can take $\overline{p}_0$ such that $\overline{p}_0 \geqq \overline{p}_0A + L > \mathbf{0}$ and $\frac{1}{\overline{p}_0\omega^{Nb}}\overline{p}_0Ab + vb \leqq 1$. Indeed, we have already argued that there exists $\overline{p}_0 = \tilde{p}_0(\varsigma_0)$, which is defined by taking a sufficiently small $\varsigma_0 > 0$ in order to meet $\varsigma_0 q \leqq \varepsilon$, such that $\overline{p}_0 \geqq \overline{p}_0A + L$ and $\overline{p}_0Ab + vb \leqq 1$ hold by Lemma VI. Moreover, by the definition of $\omega^{Nb} = A(I-A)^{-1}Nb$, $\frac{1}{\overline{p}_0\omega^{Nb}}\overline{p}_0Ab < \overline{p}_0Ab$ holds.

Let $p_1^r \equiv (1+r)\overline{p}_0A + (1+r)L$ for $r \geqq 0$. For each $\theta \in [0,1]$, we define the set

$$X^r(\theta) \equiv \left\{x \in \mathbb{R}_+^n \mid \overline{p}_0Ax = \theta\overline{p}_0\omega^{Nb},\ p_1^r(I-A)x = p_1^rb,\ Lx \in (0,1)\right\}$$

with generic element $x^r(\theta)$. Note that $X^r(\theta)$ is non-empty for some $\theta \in [0,1]$, since $(I-A)^{-1}b \in X^r\left(\frac{1}{N}\right)$. Indeed, $Ax^r\left(\frac{1}{N}\right) = A(I-A)^{-1}b = \frac{1}{N}\omega^{Nb}$ holds, so that

$$\overline{p}_0Ax^r(\theta) = \overline{p}_0Ax^r\left(\frac{1}{N}\right) = \frac{1}{N}\overline{p}_0\omega^{Nb} = \theta\overline{p}_0\omega^{Nb}.$$

Also, $(I-A)x^r\left(\frac{1}{N}\right) = b$ implies $p_1^r(I-A)x^r\left(\frac{1}{N}\right) = p_1^rb$. Finally, $L(I-A)^{-1}(Nb) \in (0,N)$ implies $\frac{1}{N}L(I-A)^{-1}(Nb) \in (0,1)$, so that

$$Lx^r(\theta) = Lx^r\left(\frac{1}{N}\right) = L(I-A)^{-1}b \in (0,1).$$

Thus, $X^r(\theta) \neq \varnothing$ for some $\theta \in [0,1)$.

Note that $X^r(\theta)$ is a compact set of $\mathbb{R}^n_+$ for any $\theta \in [0,1]$ since $\overline{p}_0 > \mathbf{0}$ and $p_1^r > \mathbf{0}$. Then, we can take $l^r(\theta) \equiv \min\{Lx^r(\theta) \mid x^r(\theta) \in X^r(\theta)\}$ for each $\theta \in [0,1]$ satisfying $X^r(\theta) \neq \varnothing$, so that $l^r(\theta) \in (0,1]$. We define

$$\theta^r \equiv \inf\{\theta \in [0,1] \mid l^r(\theta) \in (0,1]\},$$

where $\theta^r > 0$ by $b > \mathbf{0}$. We also define $\underline{\theta} \equiv \inf_{r \geqq 0} \theta^r > 0$. Then, consider the set:

$$\underline{\Delta}(W_0) \equiv \left\{ (W_0^\nu)_{\nu \in \mathcal{N}} \in \mathbb{R}^N_{++} \mid \sum_{\nu \in \mathcal{N}} W_0^\nu = 1, \ \min_{\nu \in \mathcal{N}} W_0^\nu > \underline{\theta} \right\}.$$

Moreover, let $\Delta(\overline{\omega}) \equiv \{p > \mathbf{0} \mid p\overline{\omega} = 1\}$.

Theorem A.1: Let $\langle \mathcal{N}, (A, L), u, \overline{\omega} \rangle$ with $\overline{\omega} = \omega^{Nb}$ be an economy satisfying $\frac{qAe_i}{L_i} > \frac{qAe_n}{L_n}$ and $\frac{vAe_i}{L_i} \geqq \frac{vAe_n}{L_n}$ for $i = 1, \ldots, n-1$, $\frac{vAe_i}{L_i} > \frac{vAe_n}{L_n}$ for some i, and $v(I+A)b < 1$. Let $\left\{\left(x_t^*, \delta_t^*, \omega_{t+1}^*\right)\right\}_{t=1}^\infty$ satisfy $Lx_t^* = Nvb$ for all $t = 1, 2, \ldots$. Moreover, let $(W_0^\nu)_{\nu \in \mathcal{N}} \in \underline{\Delta}(W_0)$ be a profile of initial wealth endowments. Then, there exist $(p_0^*, w_0^*) \in \Delta(\overline{\omega}) \times \mathbb{R}_{++}$, $\mathbf{r}^* \subset \overline{\mathbb{R}}_+$, and $(\omega_1^{*\nu})_{\nu \in \mathcal{N}} \in \mathbb{R}^{nN}_+$ satisfying $p_0^* \omega_1^{*\nu} = W_0^\nu$ by which $\left\{\left(x_t^{*\nu}, \delta_t^{*\nu}, \omega_{t+1}^{*\nu}\right)_{\nu \in \mathcal{N}}\right\}_{t=1}^\infty$ constitutes an RS with $\left\{p_{t-1}^*; (w_t^*, r_t^*)\right\}_{t=1}^\infty = \tilde{\mathbf{p}}(p_0^*, w_0^*, \mathbf{r}^*)$.

Proof. Let $\left\{\left(x_t^*, \delta_t^*, \omega_{t+1}^*\right)\right\}_{t=1}^\infty$ satisfy $Lx_t^* = Nvb$ for all $t = 1, 2, \ldots$. Proposition A.1 implies that, for each $t = 1, 2, \ldots$, any assignment $(x_t^{*\nu})_{\nu \in \mathcal{N}} \in \mathbb{R}^{nN}_+$ satisfying $\sum_{\nu \in \mathcal{N}} x_t^{*\nu} = x_t^*$ and $Lx_t^{*\nu} \in [0,1]$ $(\forall \nu \in \mathcal{N})$ can constitute a Pareto efficient allocation. Note that $x_t^* = (I-A)^{-1}(Nb)$ and $\delta_t^* = \mathbf{0}$ for all $t = 1, 2, \ldots$. Then, x_1^* satisfies $pAx_1^* = p\omega^{Nb}$ and $px_1^* = Npb + p\omega^{Nb}$ for any $p \geqq \mathbf{0}$ by an argument similar to the above case for $x^r\left(\frac{1}{N}\right) \in X^r\left(\frac{1}{N}\right)$.

Let $(W_0^\nu)_{\nu \in \mathcal{N}} \in \underline{\Delta}(W_0)$. By the construction, there exists $r \geqq 0$ such that $\min_{\nu \in \mathcal{N}} W_0^\nu \geqq \theta^r$, and there exists $x(r) \in X^r(\theta^r)$ with $\overline{p}_0$ satisfying $\overline{p}_0 \geqq \overline{p}_0 A + L > \mathbf{0}$ and $\frac{1}{\overline{p}_0 \omega^{Nb}} \overline{p}_0 Ab + vb \leqq 1$, and $p_1^r \equiv (1+r)\overline{p}_0 A + (1+r)L$. Define $p_0^* \equiv \frac{1}{\overline{p}_0 \omega^{Nb}} \overline{p}_0 \in \Delta\left(\omega^{Nb}\right)$. Then, there exists $(\theta^\nu)_{\nu \in \mathcal{N}} \in \mathbb{R}^N_{++}$ such that $\left(\theta^\nu \omega^{Nb}\right)_{\nu \in \mathcal{N}} \in \mathbb{R}^{nN}_+$ and $W_0^\nu = \theta^\nu p_0^* \omega^{Nb}$ $(\forall \nu \in \mathcal{N})$, and thus, we can take $(\omega_1^{*\nu})_{\nu \in \mathcal{N}} \in \mathbb{R}^{nN}_+$ such that $p_0^* \omega_1^{*\nu} = W_0^\nu$ $(\forall \nu \in \mathcal{N})$. For each $\nu \in \mathcal{N}$, let $x_1^{*\nu} \in \mathbb{R}^n_+$ be a solution to:

$$\min_{x \in \mathbb{R}^n_+} Lx, \text{ s.t. } p_0^* Ax = \theta^\nu p_0^* \omega^{Nb} = p_0^* \omega_1^{*\nu}; \ p_1^*(I-A)x = p_1^* b,$$

where $p_1^* \equiv \frac{1}{\overline{p}_0 \omega^{Nb}} p_1^r = (1+r) p_0^* A + \frac{1}{\overline{p}_0 \omega^{Nb}} (1+r) L$. As we have seen in the proof of Proposition A.4, for each $t = 1, 2, \ldots$, there exist $(x_t^{*\nu})_{\nu \in \mathcal{N}}$ and $\left(x_{t+1}^{*\nu}\right)_{\nu \in \mathcal{N}} \in X$ such that $p_{t-1}^r A x_{t+1}^{*\nu} = p_{t-1}^r A x_t^{*\nu}$ and $p_t^r (I - A) x_t^{*\nu} = p_t^r b$ for $\left\{p_{t-1}^r; (w_t, r_t^*)\right\}_{t=1}^{\infty} = \mathbf{p}(p_0^r, 1, \mathbf{r}^*)$, where $p_0^r \equiv \overline{p}_0$ and $\mathbf{r}^* \subset \overline{\mathbb{R}}_+$ starts with $r_1^* = r$. Then, taking $\left\{p_{t-1}^*; (w_t^*, r_t^*)\right\}_{t=1}^{\infty} = \mathbf{p}(p_0^*, w_0^*, \mathbf{r}^*)$ for $w_0^* \equiv \frac{1}{\overline{p}_0 \omega^{Nb}} > 0$, we obtain $\left\{(x_t^{*\nu})_{\nu \in \mathcal{N}}\right\}_{t=1}^{\infty} \subset X$ such that $p_0^* A x_1^{*\nu} = p_0^* \omega_1^{*\nu}$, $p_t^* A x_{t+1}^{*\nu} = p_t^* A x_t^{*\nu}$, and $p_t^* (I - A) x_t^{*\nu} = p_t^* b$ for all $t = 1, 2, \ldots$. Let $\left\{\left(\delta_t^{*\nu}, \omega_{t+1}^{*\nu}\right)_{\nu \in \mathcal{N}}\right\}_{t=1}^{\infty}$ consist of $\delta_t^{*\nu} = \mathbf{0}$ and $\omega_{t+1}^{*\nu} \equiv A x_t^{*\nu}$ $(\forall \nu \in \mathcal{N}, \forall t = 1, 2, \ldots)$. Then, $\left\{\left(x_t^{*\nu}, \delta_t^{*\nu}, \omega_{t+1}^{*\nu}\right)_{\nu \in \mathcal{N}}\right\}_{t=1}^{\infty}$ constitutes an RS with $\left\{p_{t-1}^*; (w_t^*, r_t^*)\right\}_{t=1}^{\infty}$ by Proposition A.4 again. □

Remark A.1: Note that in Theorem A1, there is no constraint on $\mathbf{r}^* \subset \overline{\mathbb{R}}_+$ except for the initial interest rate r_1^*. Therefore, this theorem allows the existence of an RS in which $\liminf_{t \to \infty} r_t^* > R$ holds.